Curvature lines on orthogonal surfaces of $\mathbb{R}^3$
and Joachimsthal Theorem

Ronaldo A. Garcia∗†
Instituto de Matemática e Estatística
Universidade Federal de Goiás Brazil

Abstract

In this paper is studied, as a complement of Joachimsthal theorem, the behavior of curvature lines near a principal cycle common to two orthogonal surfaces.

keywords: principal cycle, curvature lines. MSC: 53C12, 34D30, 53A05, 37C75

1 Introduction

The local behavior of curvature lines near umbilic points was considered by G. Darboux, [3], for analytic surfaces and by C. Gutierrez and J. Sotomayor, [7], for $C^r$ surfaces.
Near principal cycles, the local behavior of curvature lines was first considered in details by C. Gutierrez and J. Sotomayor, [7]. They obtained the derivative of the first return map $\pi : \Sigma \to \Sigma$ associated to the periodic leaf and showed that generically (open and dense set of immersions) the principal cycles are hyperbolic, i.e, $\pi'(0) \neq 1$.
The Joachimsthal theorem says that two surfaces intersecting at a constant angle along a regular curve $\gamma$ and this curve is a curvature line of one surface then it is a curvature line of the other.
The main goal of this paper is to describe the local behavior near a principal cycle common to two surfaces intersecting orthogonally.

∗ragarcia@mat.ufg.br
†The author was partially supported by FUNAPE/UFG and is fellow of CNPq. This work was done under the project PRONEX/FINEP/MCT - Conv. 76.97.1080.00 - Teoria Qualitativa das Equações Diferenciais Ordinárias and had the partial support of CNPq Grant 476886/2001-5.
2 Differential equation of curvature lines

A principal curvature line is a regular curve (parametrized by arc length) \( \gamma : (a, b) \to \mathbb{M} \setminus U \) such that for all \( s \in (a, b) \) we have \( \gamma'(s) \) is a principal direction.

The normal curvature at \( p \) in the direction \( w \in T_p\mathbb{M} \) is \( k_n(p; w) = II(p; w)/I(p; w) \), where \( I \) and \( II \) are, respectively, the first and second fundamental forms of \( \mathbb{M} \).

Therefore, \( w = (du, dv) \) is a principal direction, if and only if, there exists \( \lambda \in \mathbb{R} \) such that

\[
II(p; w) = \lambda I(p; w), \quad I(p; w) = 1.
\]

This means that \( I \) e \( II \) are proportional in the direction \( w \).

As \( I(p; w) = Edu^2 + 2Fdudv + Gdv^2 \) and \( II(p; w) = edu^2 + 2fdudv + gdv^2 \) we have that \( w = (du, dv) \) is a principal direction, if and only if,

\[
\frac{\partial(I, II)}{\partial(du, dv)} = 0.
\]

Or, equivalently by,

\[
(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0. \tag{1}
\]

In the case where \( \mathbb{M} \) is parametrized as graph \( (x, y, h(x, y)) \) we have that

\[
E = 1 + h_x^2, \quad F = h_x h_y, \quad G = 1 + h_y^2, \quad e = \frac{h_{xx}}{\sqrt{EG - F^2}}, \quad f = \frac{h_{xy}}{\sqrt{EG - F^2}}, \quad g = \frac{h_{yy}}{\sqrt{EG - F^2}}.
\]

When \( \mathbb{M} \) is defined implicitly \( \mathbb{M} = \{(x, y, z) : h(x, y, z) = 0\} \) the differential equation of curvature lines is expressed by

\[
[dp, \nabla h, d\nabla h] = 0,
\]

where \( dp = (dx, dy, dz), \nabla h = (h_x, h_y, h_z), \) \( d\nabla h = (dh_x, dh_y, dh_z) \) and \([,\,,\,] \) denotes the mist product of three vectors.

Remark 1. See the books and lecture notes [1], [2], [5], [7], [6], [8], [9], [10], [11] and [12] for more on local and global properties of principal curvature lines on surfaces.

3 General properties of curvature lines

Theorem 1 (Joachimsthal). Let \( \mathbb{M}_1 \subset \mathbb{R}^3 \) and \( \mathbb{M}_2 \subset \mathbb{R}^3 \) two regular and oriented surfaces such that \( \mathbb{M}_1 \cap \mathbb{M}_2 = \gamma \) is a regular curve and \( \langle N_1(\gamma(s)), N_2(\gamma(s)) \rangle = cte \) along \( \gamma \), where \( N_1 \) and \( N_2 \) are unitary normal vector fields to \( \mathbb{M}_1 \) and \( \mathbb{M}_2 \). Then \( \gamma \) is a principal curvature line of \( \mathbb{M}_1 \) if and only if it is a curvature line of \( \mathbb{M}_2 \).
Proof. Suppose that $\langle N_1(\gamma(s)), N_2(\gamma(s)) \rangle = 0$.
Let $T = \gamma'(s)$ and suppose that $\gamma$ is a principal curvature line, with geodesic curvature $k_{g,1}$, geodesic torsion $\tau_{g,1} = 0$ and principal curvature $k_{m,1}$, for the surface $\mathbb{M}_1$. See [11]. So,

$$T' = k_{g,1}N_1 \wedge T + k_{m,1}N_1$$

$$(N_1 \wedge T)' = -k_{g,1}T + \tau_{g,1}N$$

$$(N_1)' = -k_{m,1}T - \tau_{g,1}N \wedge T$$

(2)

The Darboux frame for $\gamma$, as a curve of $\mathbb{M}_2$, is given by:

$$T' = k_{g,2}N_2 \wedge T + k_{n,2}N_2$$

$$(N_2 \wedge T)' = -k_{g,2}T + \tau_{g,2}N_2$$

$$(N_2)' = -k_{n,2}T - \tau_{g,2}(N_2 \wedge T)'$$

(3)

where $k_{n,2}$ is the normal curvature, $\tau_{g,2}$ is the geodesic torsion and $k_{g,2}$ is the geodesic curvature of $\gamma$ as a curve of $\mathbb{M}_2$.

Also $N_2 = \pm N_1 \wedge T$, since $\langle N_1, N_2 \rangle = 0$. Suppose $N_2 = N_1 \wedge T$. From the equations (2) and (3), and using that $N_1 = T \wedge N_2$, it follows that:

$$\tau_{g,2} = \tau_{g,1} = 0$$

$$k_{g,1} = k_{m,2}$$

$$k_{g,2} = k_{m,1}$$

where $k_{m,2}$ is a principal curvature of $\mathbb{M}_2$. Therefore $\gamma$ is a principal curvature line of $\mathbb{M}_2$.

The case $\langle N_1, N_2 \rangle = cte \neq 0$ is analogous.

**Proposition 1.** A closed, simple and biregular curve $c : \mathbb{R} \to \mathbb{R}^3$, $\|c'(s)\| = 1$, of length $L$ and torsion $\tau$ is a principal curvature line of a surface if, and only if, $\int_0^L \tau(s)ds = 2k\pi$, $k \in \mathbb{N}$.

**Proof.** Consider the Frenet frame $\{t, n, b\}$ associated to $c$.
Let $N = \cos \theta(s)n(s) + \sin \theta(s)b(s)$ be a unitary normal vector to $c$.
So it follows that,

$$N'(s) = -k(s)\cos \theta(s)t(s) + (\theta'(s) + \tau(s))[-\sin \theta(s)n(s) + \cos \theta(s)b(s)].$$

Therefore, $N'(s) = \lambda t(s)$ if and only if $\theta'(s) + \tau(s) = 0$.
So $\theta(L) - \theta(0) = -\int_0^L \tau(s)ds \in N(L) = N(0)$ if and only if $\int_0^L \tau(s)ds = 2k\pi$, $k \in \mathbb{N}$.

**Proposition 2.** Let $\gamma : [0, L] \to \mathbb{R}^3$ be a principal cycle of a surface $\mathbb{M}$ such that $\{T, N \wedge T, N\}$ is a positive frame of $\mathbb{R}^3$. Then the expression

$$\alpha(s, v) = \gamma(s) + v(N \wedge T)(s)$$

$$+ \left( \frac{1}{2}k_2(s)v^2 + \frac{1}{6}b(s)v^3 + \frac{1}{24}c(s)v^4 + o(v^4) \right) N(s), \ -\delta < v < \delta$$

(4)
where $k_2$ is the principal curvature in the direction of $N \wedge T$, defines a local $C^\infty$ chart on the surface $\hat{M}$ defined in a small tubular neighborhood of $\gamma$.

**Proof.** The map $\alpha(s,v,w) = c(u) + v(N \wedge T)(s) + wN(s)$ is a local diffeomorphism in a neighborhood of the $s$ axis. For each $s$, the curve $v \to v(N \wedge T)(s) + w(s,v)N(s)$ is the intersection of the surface $\hat{M}$ with the plane spanned by $\{(N \wedge T)(s), N(s)\}$. Using Hadamard’s lemma it follows that

$$w(s,v) = \left[\frac{1}{2}k_2(s)v^2 + v^2A(s,v)\right]N(s)$$

where $A(s,0) = 0$ and $k_2$ is the (plane) curvature of the curve in the plane spanned by $\{N \wedge T, N\}$, that cuts the surface $\hat{M}$. This ends the proof. 

According to [11], the Darboux frame $\{T, N \wedge T, N\}$ along $\gamma$ satisfies the following system of differential equations:

\[
\begin{align*}
T' &= k_gN \wedge T + k_1N \\
(N \wedge T)' &= -k_gT + 0N \\
N' &= -k_1T - 0(N \wedge T)
\end{align*}
\]

(5)

where $k_1$ is the principal curvature and $k_g$ is the geodesic curvature of the principal cycle $\gamma$.

### 4 Preliminary calculations

Consider the parametrizations $\alpha$ of $M_1$ and $\beta$ of $M_2$ in a neighborhood of $\gamma$, such that $\{T, N \wedge T, N\}$ is a positive frame of $\gamma$ as a curve of $M_1$ and $\{T, N, T \wedge N\}$ is a positive frame of $\gamma$ as a curve of $M_2$.

\[
\begin{align*}
\alpha(s,v) &= \gamma(s) + v(N \wedge T)(s) + \left[\frac{1}{2}k_2(s)v^2 + \frac{1}{6}b(s)v^3 + O(v^3)\right]N(s) \\
\beta(s,w) &= \gamma(s) + wN(s) + \left[\frac{1}{2}m_2(s)w^2 + \frac{1}{6}B(s)w^3 + O(w^3)\right](T \wedge N)(s)
\end{align*}
\]

(6)

### 4.1 Immersion $\alpha$

The coefficients of the first fundamental form of $\alpha$ are given by:

\[
\begin{align*}
E_\alpha(s,v) &= 1 - 2k_gv + [k_g^2 - k_1k_2]v^2 + O(v^3) \\
F_\alpha(s,v) &= O(v^3) \\
G_\alpha(s,v) &= 1 + k_2^2v^2 + O(v^3)
\end{align*}
\]

(7)

The unitary normal vector field $N_\alpha = (\alpha_s \wedge \alpha_v)/|\alpha_s \wedge \alpha_v|$ is given by:

144
The functions \(N_\alpha(s, v)\) are given by:

\[
N_\alpha(s, v) = \left[ -\frac{1}{2}k_2^2v^2 + O(v^3) \right]T(s) - \left[ k_2v + \frac{1}{2}b(s)v^2 + O(v^3) \right](N \wedge T)(s)
+ \left[ 1 - \frac{1}{2}k_2^2v^2 + O(v^3) \right]N(s)
\]  

(8)

The coefficients of the second fundamental form of \(\alpha\) are given by:

\[
e_\alpha(s, v) = k_1 - (k_1 + k_2)k_gv
+ \frac{1}{2}[k_2'' - (k_1 + k_2)k_1k_2 - k_gb(s) + 2k_2^2k_2v^2 + O(v^3)]
\]

\[
f_\alpha(s, v) = k_2'v + \frac{1}{2}[k_gk_2' + b'(s)]v^2 + O(v^3)
\]

\[
g_\alpha(s, v) = k_2 + b(s)v + \frac{1}{2}(c(s) - k_2^2)v^2 + O(v^3)
\]

(9)

The functions \(L_\alpha = (Fg - Gf)_\alpha\), \(M_\alpha = (Eg - Ge)_\alpha\) and \(N_\alpha = (Ef - Fe)_\alpha\) are given by:

\[
L_\alpha(s, v) = -k_2'v - \frac{1}{2}(k_2k_2' + b'(s))v^2 + O(v^3)
\]

\[
M_\alpha(s, v) = k_2 - k_1 + [k_1 - k_2]k_g + b(s)]v
+ \frac{1}{2}[-3k_2k_2 - 3k_gb(s) + c(s) - k_2^2 - k_2' + k_2k_2]v^2 + O(v^3)
\]

\[
N_\alpha(s, v) = k_2'v + \frac{1}{2}(b'(s) - 3k_gk_2)v^2 + O(v^3)
\]

(10)

The functions \(K_\alpha\) and \(H_\alpha\) are given by:

\[
K_\alpha(s, v) = k_1k_2 + [(k_1k_2 - k_2^2)k_g(s) + k_1b(s)]v + O(v^2)
\]

\[
H_\alpha(s, v) = \frac{1}{2}(k_2k_2 + 1) + \frac{1}{2}[k_1 - k_2]k_g + b(s)]v + O(v^2)
\]

(11)

The principal curvatures \(k_{1,\alpha} = H_\alpha - \sqrt{H_\alpha^2 - K_\alpha}\) and \(k_{2,\alpha} = H_\alpha + \sqrt{H_\alpha^2 - K_\alpha}\) are given by:

\[
k_{1,\alpha}(s, v) = k_1 + (k_1 - k_2)k_gv + O(v^2)
\]

\[
k_{2,\alpha}(s, v) = k_2 + b(s)v + O(v^2)
\]

(12)

Remark 2. The following relations holds

\[
k_g(s) = -\frac{(k_1)v}{k_2 - k_1},\quad k_g^+(s) = -\frac{(k_2)'v}{k_2 - k_1},\quad b(s) = (k_2)_v = \frac{\partial k_2}{\partial v}
\]

(13)

Here \(k_g^+(s)\) is the geodesic curvature of the other principal curvature line which pass through \(\gamma(s)\).
4.2 Immersion $\beta$

The coefficients of the first fundamental form of $\beta$ are given by:

\[
E_\beta(s, w) = 1 - 2k_1w + (k_1^2 + k_g m_2)w^2 + O(w^3)
\]
\[
F_\beta(s, w) = O(w^3)
\]
\[
G_\beta(s, w) = 1 + m_2^2w^2 + O(w^3)
\]  \hfill (14)

The unitary normal vector field $N_\beta = \beta_s \wedge \beta_w/|\beta_s \wedge \beta_w|$ is given by:

\[
N_\beta(s, w) = [- \frac{1}{2} m'_2w^2 + O(w^3)]T(s) - [m_2w + \frac{1}{2}B(s)w^2 + O(w^3)](N \wedge T)(s)
+ [1 - \frac{1}{2} m_2^2w^2 + O(w^3)]N(s)
\]  \hfill (15)

The coefficients of the second fundamental form of $\beta$ are given by:

\[
e_\beta(s, w) = -k_g - k_1[m_2 - k_g]w
+ \frac{1}{2}[m''_2 - k_1 B(s) + 2k_1^2 m_2 + k_g^2 m_2 + k_1 m_2^2]w^2 + O(w^3)
\]
\[
f_\beta(s, w) = m'_2v + \frac{1}{2}[k_1 m'_2 + B'(s)]w^2 + O(w^3)
\]
\[
g_\beta(s, w) = m_2 + B(s)w + \frac{1}{2}(C(s) - m_2^2)w^2 + O(w^3)
\]  \hfill (16)

The functions $L_\beta = (Fg - Gf)_\beta$, $M_\beta = (Eg - Ge)_\beta$ and $N_\beta = (Ef - Fe)_\beta$ are given by:

\[
L_\beta(s, w) = -m'_2w - \frac{1}{2}(k_1 m'_2 + B'(s))w^2 + O(w^3)
\]
\[
M_\beta(s, w) = m_2 + k_g + [B(s) - k_1(m_2 + k_g)]v
+ \frac{1}{2}[(3k_g m_2^2 - 3k_1 B(s) + C(s) - m_2^2 - m_2'' - k_g^2 m_2^2]w^2 + O(w^3)
\]
\[
N_\beta(s, w) = m'_2(s)v + \frac{1}{2}(B'(s) - 3k_1 m'_2)w^2 + O(w^3)
\]  \hfill (17)

The functions $K_\beta$ and $H_\beta$ are given by:

\[
K_\beta(s, w) = -k_g m_2 - [(k_g m_2 + m_2^2)k_1 + k_g B(s)]w + O(w^2)
\]
\[
H_\beta(s, w) = \frac{1}{2}(m_2 - k_g) + \frac{1}{2}[B(s) - (k_g + m_2)k_1]w + O(w^2)
\]  \hfill (18)

The principal curvatures $k_{1,\beta} = H_\beta - \sqrt{H_\beta^2 - K_\beta}$ and $k_{2,\beta} = H_\beta + \sqrt{H_\beta^2 - K_\beta}$ are given by:

\[
k_{1,\beta}(s, w) = -k_g - (k_g + m_2)k_1w + O(w^2)
\]
\[
k_{2,\beta}(s, w) = m_2 + B(s)w + O(w^2)
\]  \hfill (19)
5 Principal cycles

Proposition 3 (Gutierrez-Sotomayor). Let $\gamma$ be a principal cycle of an immersion $\alpha : \mathcal{M} \rightarrow \mathbb{R}^3$ of length $L$. Denote by $\pi_\alpha$ the first return map associated to $\gamma$. Then

$$
\pi_\alpha'(0) = \exp \left[ \int_{\gamma} -\frac{dk_2}{k_2 - k_1} \right] = \exp \left[ \int_{\gamma} \frac{1}{k_1} (s) ds \right] = \exp \left[ \int_{\gamma} \frac{1}{2} \int_{\gamma} \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - k}} \right].
$$

(20)

Proof. Suppose that $\gamma$ is a principal cycle and consider the chart $(s, v)$ as defined by the expression of $\alpha$ in the equation (6). The differential equation of the principal curvature lines is given by

$$(f - k_1F)ds + (g - k_1G)dv = 0.$$  

(21)

Therefore $\pi(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of equation 21 with initial condition $v(0, v_0) = v_0$.

Differentiation of equation 21 with respect to $v_0$ gives:

$$
\frac{d}{ds} \left( \frac{\partial v}{\partial v_0} \right)(s, v(s, v_0)) = \left[ f - k_1F \right] g - k_1G \partial v \partial v_0 (s, v(s, v_0)).
$$

Denote $a(s) = \left( \frac{\partial v}{\partial v_0} \right)(s, 0)$. Therefore at $v(s, 0) = 0$ it is obtained

$$
\frac{d}{ds} a(s) = -f_v(s, 0) a(s) = -\frac{k_1}{g - k_1} a(s) = k_2^{-2} a(s), \quad a(0) = 1.
$$

Integration of the linear differential equation above leads to the result. 

The following result established in [4] is improved in the next proposition.

Proposition 4. Let $\gamma$ be a principal cycle of length $L$ of a surface $\mathcal{M} \subset \mathbb{R}^3$. Consider a chart $(s, v)$ and a parameterization $\alpha$ as defined by equation (6). Denote by $k_1$ and $k_2$ the principal curvatures of $\mathcal{M}$. Suppose that $\text{Jac}(k_1, k_2) = \frac{\partial (k_1, k_2)}{\partial (s, v)} = (k_1)_s(k_2)_v - (k_1)_v(k_2)_s \neq 0$ for all $s \in [0, L]$. Then if $\gamma$ is not hyperbolic then it is semihyperbolic. That is, if the first derivative of the first return map $\pi$ associated to $\gamma$ is one, then the second derivative of $\pi$ is different from zero. In fact, if $\pi'(0) = 1$ then,

$$
\pi''(0) = \int_0^L e^{-\int_0^s \frac{\epsilon_2}{k_2 - k_1} du} \frac{\text{Jac}(k_1, k_2)}{(k_2 - k_1)^2} ds.
$$
Proof. The differential equation of the principal curvature lines 21 in the chart \((s, v)\) is given by
\[
\frac{dv}{ds} = -\frac{f - k_1F}{g - k_1G} = -\frac{k'_2}{k_2 - k_1}v - \frac{1}{2}\frac{b'(k_2 - k_1) - 2k'_2b + k_gk'_2(k_1 - k_2)}{(k_2 - k_1)^2}v^2 + v^2R(s, v). \tag{22}
\]
Therefore \(\pi(v_0) = v(L, v_0)\), where \(v(s, v_0)\) is the solution of equation (22) with initial condition \(v(0, v_0) = v_0\).

Differentiating twice the equation (22) with respect to \(v_0\) and evaluating at \(v_0 = 0\) the following holds
\[
\frac{d}{ds}\left(\frac{\partial v}{\partial v_0}\right) = P(s)\frac{\partial v}{\partial v_0} + Q(s)(\frac{\partial v}{\partial v_0})^2, \quad \frac{\partial v}{\partial v_0}(0) = 1, \quad \frac{\partial^2 v}{\partial v_0^2}(0) = 0.
\]

So,
\[
\pi''(0) = \left.\frac{\partial^2 v}{\partial v_0^2}\right|_0 = \int_0^L \exp\left(\int_0^s P(u)du\right)Q(s)ds = \int_0^L \exp\left(-\int_0^s \frac{k'_2}{k_2 - k_1}du\right)\frac{2k'_2b - b'(k_2 - k_1) - k_gk'_2k_1 - k_2}{(k_2 - k_1)^2}ds.
\]

Integration by parts and using that \(k_g(k_1 - k_2) = \frac{\partial k_1}{\partial v}\) it follows that
\[
\pi''(0) = \int_0^L \exp\left(-\int_0^s \frac{k'_2}{k_2 - k_1}du\right)\frac{k'_1\frac{\partial k_g}{\partial v} - k'_2\frac{\partial k_1}{\partial v}}{(k_2 - k_1)^2}ds.
\]

Proposition 5. Let \(c : \mathbb{R} \to \mathbb{R}^3, |c'(s)| = 1\) be a closed, simple and biregular curve of length \(L\) and torsion \(\tau\) such that \(\int_0^L \tau(s)ds = 2k\pi, k \in \mathbb{N}\). Then there exists an immersion \(\alpha : [0, L] \times (-\epsilon, \epsilon) \to \mathbb{R}^3\) such that \(\alpha(s, 0) = c(s)\) is a hyperbolic principal cycle of \(\alpha\).
Curvature lines on orthogonal surfaces

Proof. It follows from propositions 2 and 3 defining the principal curvatures adequately.

Theorem 2. Let \( \gamma \) be a hyperbolic (minimal) principal cycle of a surface \( M \subset \mathbb{R}^3 \) of length \( L \). Let \( k_1 \) and \( k_2 \) the principal curvatures of \( M_1 \) and \( k_2 \) the geodesic curvature of \( \gamma \). Let \( P(s) = k_2'(s)/(k_2 - k_1) \) and suppose that the linear differential equation \( f' = P(s) f + k_2' \) has a \( L \)-periodic solution such that \( f(s) \neq 0 \) for all \( s \in [0, L] \). Then there exists a surface \( M_2 \subset \mathbb{R}^3 \) such that \( \gamma \) is a principal hyperbolic principal cycle of \( M_2 \) which is orthogonal to \( M_1 \) along \( \gamma \) and \( \pi_1'(0) = \pi_2'(0) \).

Proof. Consider the parametrizations \( \alpha \) of \( M_1 \) and \( \beta \) of \( M_2 \) in a neighborhood of \( \gamma \),

\[
\alpha(s, v) = \gamma(s) + v(N \wedge T)(s) + \left[ \frac{1}{2} k_2(s)v^2 + \frac{1}{6} b(s)v^3 + O(v^3) \right] N(s)
\]

\[
\beta(s, w) = \gamma(s) + wN(s) + \left[ \frac{1}{2} m_2(s)w^2 + \frac{1}{6} B(s)w^3 + O(w^3) \right] (T \wedge N)(s).
\]

where \( \{T, N \wedge T, N\} \) is a positive frame of \( \gamma \) as curve of \( M_1 \) and \( \{T, N, T \wedge N\} \) is a positive frame of \( \gamma \) as curve of \( M_2 \).

By proposition 3 it follows that

\[
\pi_\alpha'(0) = \exp[-\int_{\gamma} \frac{dk_2}{k_2 - k_1}], \quad \pi_\beta'(0) = \exp[-\int_{\gamma} \frac{dm_2}{m_2 + k_2}].
\]  

Suppose that the following equation holds

\[
\frac{k_2'}{k_2 - k_1} = \frac{m_2'}{m_2 + k_2}.
\]

Then \( m_2 \) is a defined by the linear differential equation:

\[
m_2' - \frac{k_2'}{k_2 - k_1}m_2 - k_2 m_2 = 0, \quad m_2(0) = m_0.
\]  

The solution of the linear equation above is given by

\[
m_2(s) = e^{\int_0^s a(t)dt}[m_0 + \int_0^s e^{-\int_0^t a(u)du} k_2(t) a(t)dt],
\]

where \( a(s) = k_2'(s)/(k_2 - k_1)(s) \).

As, by hypothesis, \( \int_0^L \frac{k_2'}{k_2 - k_1} \neq 0 \) it follows that \( m_0 = m_2(0) = m_2(L) \) if and only if

\[
m_0 = \frac{\int_0^L (e^{-\int_0^t a(u)du}) k_2(t) a(t)dt}{e^{-\int_0^L k_2'(s)/k_2 - k_1} ds - 1}.
\]
Therefore the immersion $\beta$ can be constructed with $m_2$, principal curvature of $\beta$, defined by the equation 24. To finish we need to show that $m_2(s) + k_g(s) \neq 0$ for all $s \in [0, L]$ and so $\gamma$ is a principal cycle of $\beta$.

In the differential equation (24) let $f = k_g + m_2$. So it is obtained,

$$f' = \frac{k'_g}{k_2 - k_1} f + k'_g. \tag{25}$$

By the same argument above the differential equation (25) has a $L-$ periodic solution.

The points $s$ where $f(s) = 0$ correspond to umbilic points of $M_2$. Therefore $\gamma$ is a principal cycle of $M_2$ if equation (25) has a periodic solution which is different from zero for all $s \in [0, L]$.

**Remark 3.** The condition $k_g \neq cte$ is a necessary condition for existence of the surface $M_2$ as stated in the theorem 2 above.

**Theorem 3.** Let $\gamma$ be a minimal principal cycle of a surface $M_1 \subset \mathbb{R}^3$ such that $k_g|_\gamma \neq cte$. Then there exists a surface $M_2 \subset \mathbb{R}^3$ such that $\gamma$ is a principal hyperbolic principal cycle of $M_2$ which is orthogonal to $M_1$ along $\gamma$.

**Proof.** By theorem 1 we have that $-k_g$ is a principal curvature of $M_2$ having $T \wedge N$ as positive normal vector in a neighborhood of $\gamma$. Defining a non constant $L-$periodic function $m_2$ such that $m_2(s) + k_g(s) > 0$ and $\int_0^L \frac{m'_2}{m_2+k_g} ds \neq 0$ the result follows, observing that

$$\int_0^L \frac{m'_2}{m_2+k_g} ds = \int_0^L \frac{-k'_g}{m_2+k_g} ds.$$

**Theorem 4.** Let $\gamma$ be a hyperbolic (minimal) principal cycle of a surface $M \subset \mathbb{R}^3$ of length $L$. Suppose that the geodesic curvature of $\gamma$ is not constant. Then there exists a surface $M_2 \subset \mathbb{R}^3$ such that $\gamma$ is a hyperbolic principal principal cycle of $M_2$ which is orthogonal to $M_1$ along $\gamma$.

**Proof.** By theorem 1 we have that $-k_g$ is a principal curvature of $M_2$ having $T \wedge N$ as positive normal vector in a neighborhood of $\gamma$. Define a non constant $L-$periodic function $m_2$ such that $m_2(s) + k_g(s) > 0$ and $\int_0^L \frac{m'_2}{m_2+k_g} ds \neq 0$. Therefore $\gamma$ is a hyperbolic (minimal) principal cycle of $M_2$ parametrized in a neighborhood of $\gamma$ by the parametrization $\beta$. Observing that

$$\int_0^L \frac{m'_2}{m_2+k_g} ds = \int_0^L \frac{-k'_g}{m_2+k_g} ds,$$

we can define $\bar{m} = m_2 + \epsilon k'_g$ to obtain $\bar{m}$ as a maximal principal curvature of $M_2$ with $\bar{m} + k_g > 0$ and $\int_0^L \frac{\bar{m}'}{\bar{m}+k_g} ds \neq 0$ for $\epsilon$ small.
References


