The μ-polynomials of graph associahedra

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Abstract

We study two polynomials associated to a graph G that are of interest in the recent literature. The first one is the h-polynomial of the graph-associahedron of G defined by Carr and Devadoss. The second one is the μ-polynomial recently defined by González D'León and Wachs, which in the case of trees the authors conjecture that is up to sign equal to its h-polynomial. We prove a more general relation between the h- and the μ-polynomial of G which in the special case of trees proves González D'León - Wachs' conjecture. We give a new description of the μ-polynomials in terms of a family of forests that we call μ-forests. As applications of the tools developed, we compute the μ-polynomials of the families of cycle and kite-like graphs. These are related to the Narayana polynomials of type A and B. We also show that these families of polynomials are real-rooted and form interlacing sequences giving support and extending previous conjectures to a general conjecture on real-rootedness and the interlacing property of the h- and the μ-polynomials of an arbitrary graph G.

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Introduction

Polynomial invariants are popular devices to study properties of graphs. Some of the most present in the literature are the characteristic polynomial (which is the characteristic polynomial of the adjacency matrix of the graph), the chromatic polynomial ([3]), and the Tutte polynomial ([24]). However, there are other well-studied polynomial invariants in graph theory (see [18]). These polynomials can encode information about vertex degrees, proper colorings, flows, number of cycles, matchings and connectedness among others. Hence they become useful tools to distinguish graphs and to tackle relevant problems in graph theory. Some of these (as it is the case of the chromatic polynomial) have not even been found to be directly connected to a solution of the problem that where originally intended to treat. However, they have become interesting algebraic subjects of study for their own sake.

The two graph polynomials we study

In this work we study two graph polynomial invariants that have appeared in the recent literature. The first of these is the h-polynomial $h_G(t)$ of a simple polytope known as the graph-associahedron P_G, defined by Carr and Devadoss in [6], constructed out of the connected subsets of a finite graph G.

Recall that to any polytope P, its *f-polynomial* is defined by

$$f_P(t) := \sum_{k\geq 0} f_k t^k,$$

where $f_i = f_i(P)$ is the number of i-dimensional faces of P. Its *h-polynomial*, is a closely related polynomial, defined using the transformation

$$h_P(t) := f_P(t-1).$$

For a graph G on $[n]$ we can consider the collection $\mathcal{B}(G)$ of subsets $I \subseteq [n]$ such that the induced subgraph of G on the vertex set I is connected. The collection $\mathcal{B}(G)$ has a special kind of combinatorial structure known as a building set (see Section 1.2 for the exact definition). Given any building set $\mathcal{B}$ on $[n]$ we can then build a polytope $P_{\mathcal{B}}$, also known as its *nestohedron* [16, Definition 6.3]. This is the polytope in $\mathbb{R}^n$ given by the Minkowsky sum

$$P_{\mathcal{B}} := \sum_{I\in\mathcal{B}} \Delta_I,$$

where Δ_I is the convex hull of the standard basis vectors $\{\boldsymbol{e}_i \mid i \in I\}$, also known as a standard simplex. Graph-associahedra $P_G := P_{\mathcal{B}(G)}$ are precisely nestrohedra defined on graphical building sets $\mathcal{B}(G)$. The h-polynomials of nestohedra where extensively studied by Postnikov, Reiner and Williams in [16].

The second graph polynomial that we will be considering is the μ-polynomial $\mu_G(t)$ defined and studied by González D'León and Wachs in [11]. These polynomials are defined as the Möbius generating polynomials of the maximal intervals of a family of posets $\mathcal{W}\Pi_G$ that are weighted versions of the classical bond lattice Π_G of a graph G, a lattice that captures the independence (or matroidal) information of G. Even though classical bond lattices are isomorphic for all trees on the same number of vertices (so they are unable to distinguish between them), the polynomials $\mu_G(t)$ appear surprisingly to be able to distinguish between them. This is conjecturally true for all trees as stated in [11].

The following connection between $h_G(t)$ and $\mu_G(t)$ when G is a tree was conjectured in [11].

Conjecture 1 (González D'León - Wachs [11]). Let T be a tree on vertex set $[n]$ then

$$\mu_T(t) = (-1)^{n-1} h_T(t).$$

Examples of h and μ polynomials

We review several infinite families of examples of h and μ polynomials. These families include the polynomials of the family of complete graphs K_n, the path graphs P_n and the star graphs St_n, whose h-polynomials have been studied in [16] and whose μ-polynomials have been studied in [11].

In the case of the family K_n, the polytope $P_{\mathcal{B}(K_n)}$ is known as the *permutahedron*. Let $\mathfrak{S}_n$ be the set of permutations on $[n]$. It is a well-known formula (see [16]) that

$$h_{K_n}(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)} =: A_n(t),$$

known as the *Eulerian polynomials*. Let $\mathcal{T}_n$ be the set of rooted trees on $[n]$. In [10] it was shown that

$$\mu_{K_n}(t) = (-1)^{n-1} \sum_{T \in \mathcal{T}_n} t^{\operatorname{des}(\sigma)} = (-1)^{n-1} \prod_{i=1}^{n-1} ((n-i) + it) =: T_n(t),$$

the *tree Eulerian polynomials*.

For the family P_n, the polytope $P_{\mathcal{B}(P_n)}$ is a realization of the *associahedron*. Let $\mathcal{P}_n$ be the set of planar complete binary trees on n internal nodes, i.e., rooted trees where each internal node has a left and a right child. For a tree $T \in \mathcal{P}_n$ we say that an internal node is a *left child* if it is the left child of its parent. It is well-known that the

Catalan numbers $Cat(n) = \dfrac{1}{n+1}\binom{2n}{n}$ gives the cardinality of $\mathcal{T}_n$ and that the *Narayana numbers* $N_{n,k} = \dfrac{1}{k}\binom{n}{k}\binom{n}{k-1}$ gives the number of trees in $\mathcal{T}_n$ with $k+1$ left children. The h-polynomial of the associahedron has the well-known formula (see [16])

$$h_{P_n}(t) = \sum_{T \in \mathcal{P}_n} t^{\mathrm{lchild}(T)} = \sum_{k=1}^{n} N_{n,k} t^{k-1} =: \mathcal{N}_n(t),$$

the *Naryana polynomials*. It was proved in [11] that

$$\mu_{P_n}(t) = (-1)^{n-1}\mathcal{N}_n(t),$$

confirming Conjecture 1.

The polytope $P_{\mathcal{B}(St_n)}$ associated to the family St_n is known as the *stellohedron* (see [16]). A closely related family of polynomials, obtained by a simple transformation from the Eulerian polynomials, are the *binomial-Eulerian polynomials*, studied by Shareshian and Wachs in [17], and defined by

$$\tilde{A}_n(t) := 1 + t\sum_{m=1}^{n}\binom{n}{m}A_m(t).$$

In [16] is shown that

$$h_{St_n}(t) = \tilde{A}_{n-1}(t),$$

and in [11] is shown that

$$\mu_{St_n}(t) = (-1)^{n-1}\tilde{A}_{n-1}(t),$$

giving another example of Conjecture 1.

We extend this list by computing the μ-polynomials of the family C_n of cycle graphs (see Section 1.1), whose h polynomials where studied also in [15]. The polytope $P_{\mathcal{B}(C_n)}$ is known as the *cyclohedron* (see [16]). The h-polynomials $h_{C_n}(t)$ of cycle graphs where computed by Simion in [19], and are given by the *type-B Narayana polynomials*

$$h_{C_n}(t) = \sum_{k=0}^{n-1}\binom{n-1}{k}^2 t^k =: W_{n-1}(t).$$

We prove the following theorem.

Theorem 2.9. The μ-polynomial $\mu_{C_n}(t)$ associated to the cyclic graph C_n on $[n]$ is given by the formula

$$\begin{aligned}\mu_{C_n}(t) &= (-1)^{n-1}\left[n\mathcal{N}_n(t) - W_{n-1}(t)\right] \qquad (2.14)\\ &= (-1)^{n-1}\sum_{k=0}^{n-1}\left[\frac{n}{k+1}\binom{n}{k}\binom{n-1}{k} - \binom{n-1}{k}^2\right]t^k.\end{aligned}$$

Relations between the h and μ polynomials

Theorem 2.9 sheds a light on a more general relation that could exists between the h and μ polynomials. Recall that $h_{C_n}(t) = W_{n-1}(t)$, $h_{P_n}(t) = \mathcal{N}_n(t)$, and there are precisely n spanning trees of C_n which are isomorphic to P_n. The only other full spanning subgraph of C_n is itself. This information is present in the formula for $\mu_{C_n}(t)$ above.

In this work we prove then a more general result that in turns implies Conjecture 1.

Theorem 3.3.

$$\mu_G(t) = \sum_{H \preceq G} (-1)^{|E(H)|} h_H(t), \tag{3.3}$$

where the sum is taken over all full spanning subgraphs H of G.

When $G = T$ is a tree on $[n]$, it has $|E(T)| = n - 1$ edges and is in itself its only full spanning subgraph. Therefore, Theorem 3.3 implies then Conjecture 1.

We use Möbius inversion, on the poset $\mathcal{G}$ whose elements are finite graphs and whose order relation is given by the full spanning relation (see Section 3.3), to prove the following formula translating back from μ-polynomials to h-polynomials.

Theorem 3.8.

$$h_{\mathcal{B}(G)}(t) = \sum_{H \preceq G} (-1)^{|E(H)|} \mu_H(t),$$

where the sum is taken over all full spanning subgraphs H of G.

Note that formulas (3.3) and (3.8) imply that the transformation that maps h-polynomials to μ-polynomials is an involution.

μ forests

In [16] the authors gave an expression for the h polynomial of any building set $\mathcal{B}$ in terms of an associated combinatorial family of trees, known as the family of $\mathcal{B}$-forests (see Section 3.1).

We introduce a related family of objects which we call *μ-forests* (See Chapter 4 for the definition) and prove the following theorem. Let $k(G)$ be the number of connected components of a graph G, and $\operatorname{des}(F)$ the number of children in a forest F with smaller labels than their parents.

Theorem 4.9. If G is a finite graph on $[n]$, the $\mu(G)$-polynomial is given by the formula

$$\mu_G(t) = (-1)^{V(G)-k(G)} \cdot \sum_{T \ \mu(G)\text{-forest}} t^{\operatorname{des} T}, \tag{4.1}$$

where the sum is taken over all μ-forests of G.

As an application of Theorem 4.9 we compute the μ-polynomials of a family that we call the kite-like graphs, denoted as U_n (see Figure 4.9).

Theorem 4.11. We have that

$$(-1)^{n-1}\mu_{U_n}(t) = 2\mathcal{N}_n(t) - t\mathcal{N}_{n-2}(t), \qquad n \geq 3, \tag{4.4}$$

where $\mathcal{N}_n(t)$ are the Narayana polynomials.

Real-rootedness and the interlacing property

Numerical exploration in [11] has suggested that the μ-polynomials appear to be real-rooted. In fact the authors propose the following conjecture.

Conjecture 2 (González D'León - Wachs [11]). For every graph G on $[n]$, the associated polynomial $\mu_G(t)$ is real-rooted. Moreover, if G is connected, then $\mu_G(t)$ has $n-1$ simple roots.

The conjecture is known to be true for the families K_n, P_n and St_n. A new result in this direction is given for the family C_n. In fact, we prove a stronger result that involves the concept of interlacing polynomials.

Given two polynomials $P, Q \in \mathbb{R}[t]$ that are real-rooted, with roots $r_1 \geq r_2 \geq \ldots$ and $s_1 \geq s_2 \geq \ldots$ respectively, we say that Q *interlaces* P if the roots satisfy

$$r_1 \geq s_1 \geq r_2 \geq \cdots .$$

If all the inequalities are strict we say that Q *strictly interlaces* P. Note that a pair of polynomials that are strictly interlacing have also simple roots.

We prove the following theorem.

Theorem 5.8. The family of polynomials $\mu_{C_n}(t)$ are real-rooted and forms an strictly interlacing sequence, that is $\mu_{C_n}(t)$ strictly interlaces $\mu_{C_{n+1}}(t)$ all $n \geq 1$, and hence the roots are also simple.

We prove an analogous result for the family U_n. Computational evidence suggest that $\mu_{U_n}(t)$ strictly interlaces $\mu_{U_{n+1}}(t)$. We can prove however a slightly related result.

Theorem 5.9. The family of polynomials $\mu_{U_n}(t)$ are real-rooted for $n \geq 1$. Moreover, $\mu_{P_n}(t)$ strictly interlaces $\mu_{U_{n+1}}(t)$ for all $n \geq 1$.

Note that the graph P_n can be obtained from U_{n+1} by the operation of contraction of one edge. Particular examples (see Section 5.3) show that not any contraction provides the strictly interlacing property. We offer the following conjecture.

Conjecture 3. For any connected graph G there exist a one-edge contraction G' of G such that $\mu_{G'}(t)$ strictly interlaces $\mu_G(t)$. As a consequence $\mu_G(t)$ has only simple real roots for any finite connected graph G.

Theorems 5.8 and 5.9 give evidence to Conjecture 3. Moreover, the families $\mu_{K_n}(t)$ (see Example 5.1.2), $\mu_{P_n}(t)$ [5] and $\mu_{St_n}(t)$ [12] have been shown to be strictly interlacing, providing more evidence for the conjecture. Computational evidence for edge contractions for all graphs up to $n = 6$ vertices also supports the interlacing part of the conjecture. The real-rootedness has been checked, however, for graphs up to $n = 7$ vertices.

A similar conjecture can be made for the h polynomials. In this case however, we have not found a particular graph G and one-edge contraction G' where the strictly interlacing property does not occur, which suggest that for h-polynomials the property could be even stronger.

Conjecture 4. For any connected graph G, any one-edge contraction G' of G satisfies that $h_{G'}(t)$ strictly interlaces $h_G(t)$. As a consequence $h_G(t)$ has only simple real roots for any finite connected graph G.

Finally, apart from these conjectures, some conclusions and questions are left open for the interested reader in Chapter 6.

1
Preliminaries

The goal of this chapter is to introduce the basic objects and tools that will be used along the text. These include graphs, rooted forests, building sets and the Möbius inversion formula for finite posets.

1.1 Graphs and trees

Let $\mathbb{N}^+$ be the set of positive integers. Fixing $n \in \mathbb{N}^+$, we will write $[n] := \{1, \dots, n\}$. If A is a set, $|A|$ will denote its cardinal.

Recall that a *(simple) graph* is a pair $G = (V, E)$ where V is a set (the *vertices* of G) and $E \subseteq \{e \subseteq V \mid |e| = 2\}$ (the *edges* of G). Two vertices $i, j \in V$ are called *incident* if $\{i, j\} \in E$. We say that $H = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

Along the text we will only work with simple graphs on $V = [n] := \{1, \dots, n\}$, for some $n \in \mathbb{N}^+$. In Figure 1.1 there are graphs on vertex set $V = [4]$. These are part of larger families of graphs on $[n]$ described below.

Example 1.1.1 (Complete graph). The complete graph K_n is defined by the set of edges

$$E(K_n) = \{\{i, j\} \mid 1 \leq i < j \leq n\}.$$

Thus, this is a simple graph having all possible edges.

Example 1.1.2 (Path graphs). This graph, denoted by P_n and also called the linear graph, has as edges the set

$$E(P_n) = \{\{i, i + 1\} \mid i = 1, \dots, n - 1\}.$$

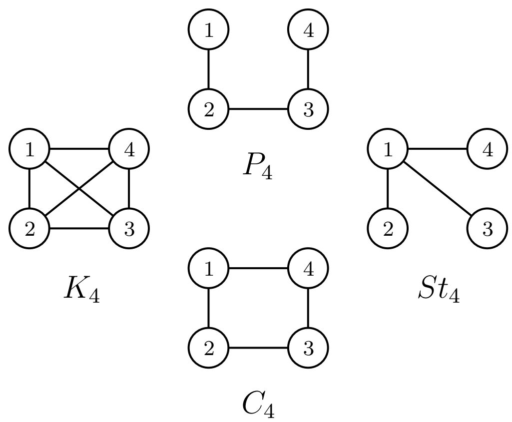

Figure 1.1: Complete K_4, path P_4, star St_4, and cycle C_4 graphs on $[4]$.

It can be seen as a line passing through all vertices.

Example 1.1.3 (Star graphs). This graph St_n is defined by

$$E(St_n) = \{\{1, i\} \mid i = 2, \ldots, n\}$$

and can be pictured as a star centered at 1 with rays emanating towards the remaining nodes.

Example 1.1.4 (Cycle graphs). The graph C_n represents a cycle joining all vertices, in such a way that

$$E(C_n) = \{\{i, i+1\} \mid i = 1, \ldots, n-1\} \cup \{\{1, n\}\}.$$

It differs from P_n only in the edge $\{1, n\}$.

A *path* L in a graph G is a collection of edges $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{m-1}, i_m\} \in E(G)$. In this case we say that L connects i_1 with i_m in G. Moreover, if $m \geq 3$ and $i_m = i_1$, L is called a *cycle* of G. It is clear that the relation $i \sim_G j$ on $[n]$ holding when i and j are connected by a path in G is an equivalence relation. A graph G is *connected* if there is only one such equivalence class, i.e., any pair of vertices can be connected by a path in G. In the previous examples all graphs K_n, P_n, St_n and C_n are connected, for all $n \geq 1$. In general, the equivalent classes $I_1 \cup \cdots \cup I_m = [n]$ decompose G into connected components $G|_{I_1}, \ldots, G|_{I_m}$. The number

$$m = k(G)$$

is the *number of connected components* of G.

A subgraph H of a graph G is said to be *spanning* in G if $V(H) = V(G)$. A spanning subgraph H of G is said to be *full spanning* if in addition we have $k(H) = k(G)$. In Figure 1.2 we see an example of a full spanning graph H of a graph G on $[8]$.

Example 1.1.5. For the path graph P_n, the only full spanning subgraph is itself. For the cyclic graph C_n, there are $n + 1$ full spanning subgraphs. One is C_n, while the remaining n are isomorphic to P_n and are obtained by removing a single edge from C_n.

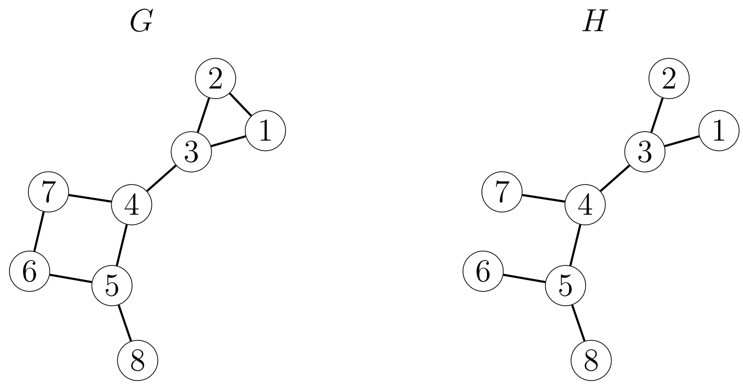

Figure 1.2: An example of a graph and a full spanning subgraph.

Other usual constructions with graphs are as follows.

Example 1.1.6 (Restrictions). If G is a graph on $[n]$ and $\emptyset \neq I \subseteq [n]$, we can consider a new graph by restricting the vertices to I: the *restriction* $G|_I$ of G to I is the graph with $V(G|_I) = I$ and

$$E(G|_I) = \{\{i,j\} \in E(G) \mid i,j \in I\}.$$

For instance, $K_n|_{[n-1]} = K_{n-1}$ and $P_n|_{[n-1]} = P_{n-1}$. However, for the cyclic graph C_n, we obtain $C_n|_{[n-1]} = P_{n-1}$.

Example 1.1.7 (Disjoint union). If $G = (V, E)$ and $G' = (V', E')$ are graphs, where $V \cap V' = \emptyset$, then their *disjoint union* is defined by

$$G \cup G' := (V \cup V', E \cup E').$$

Note that in this situation $k(G \cup G') = k(G) + k(G')$. Moreover, the spanning (full spanning) subgraphs of $G \cup G'$ have the form $H \cup H'$, where H and H' are spanning (full spanning) subgraphd of G and G', respectively.

There is another fundamental operation that we will require later obtained by contracting an edge of a graph. Intuitively, this procedure removes an edge from a graph while simultaneously merging the two vertices that it previously joined. Formally, we have the following construction.

Example 1.1.8 (Contractions). Let $G = (V, E)$ be a graph and $e = \{i,j\} \in E$. Consider $V' = (V \setminus \{i,j\}) \cup \{i'\}$, where i' is a new vertex. The *contraction of G by e* is the graph

$$G/e := (V', E'), \qquad E' = \{\{k,l\} \in E \mid k,l \neq i,j\} \cup \{\{i',k\} : \{i,k\} \in E \text{ or } \{j,k\} \in E\}.$$

Note that $|V(G/e)| = |V(G)| - 1$, and if G is connected, the same is true for G/e. More generally, $k(G/e) = k(G)$, for any $e \in E(G)$.

In Figure 1.3 we see an example of a graph G on $[6]$ contracted by the edge $\{1, 3\}$.

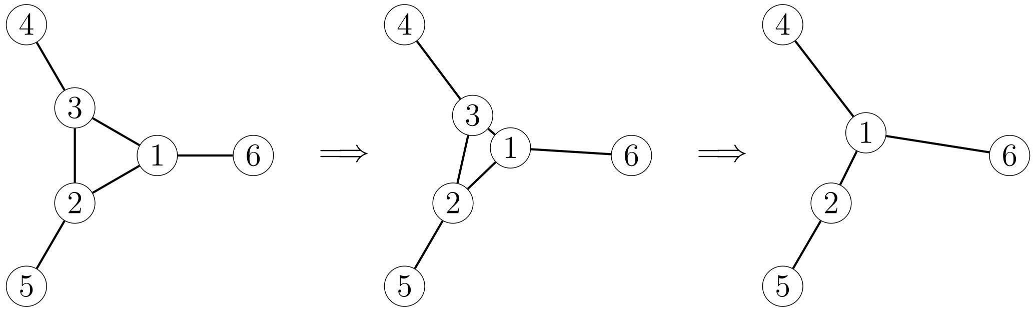

Figure 1.3: An example of a contraction.

Example 1.1.9. When we contract K_n by any edge we obtain a graph isomorphic to K_{n-1}. The same is true for P_n, C_n, and St_n, obtaining graphs isomorphic to P_{n-1}, C_{n-1}, and St_{n-1}, respectively.

To conclude this section we recall a special type of graphs F having no cycles. These are known as *forests*. If F is connected and has no cycles, then F is called a *tree*. Therefore, a forest consists of a finite number of trees.

There are several characterizations of forests. For instance, F is a forest if and only if for different $i, j \in V(F)$, there is at most one path in F from i to j.

A *rooted forest* is a forest where in each connected component (tree) a distinguished node (a *root*) has been selected. In particular, for a rooted tree T, its edges can be assigned a natural orientation, either away from or towards the root. Intuitively, by choosing a root τ of a tree we lift the graph from τ in such a way that their children are placed below it, and the same holds recursively for the remaining vertices.

For a rooted forest F, given two nodes i and j of F, we say that i is a *descendant* of j if j belongs to the shortest path connecting i with the root of its connected component. We denote by $F_{\leq j}$ the set of descendants of j. Note that in particular, $j \in F_{\leq j}$. A descendant i of j is called a *child* of j (denoted by $i \lessdot j$) if i is a descendant of j and $\{i, j\}$ is an edge of F. Finally, we say that i and j are *incomparable* if neither i is a descendant of j, nor j is a descendant of i.

Example 1.1.10. The following is an example of a tree T on $[10]$ and a rooted tree obtained from T with root at 2. The children of 2 are 1 and 5 which are placed in the second level of the tree. Also $3 \lessdot 1$, $4 \lessdot 1$, $6 \lessdot 5$, $8 \lessdot 5$, and so on. For instance, $T_{\leq 5} = \{5, 6, 7, 8, 9, 10\}$.

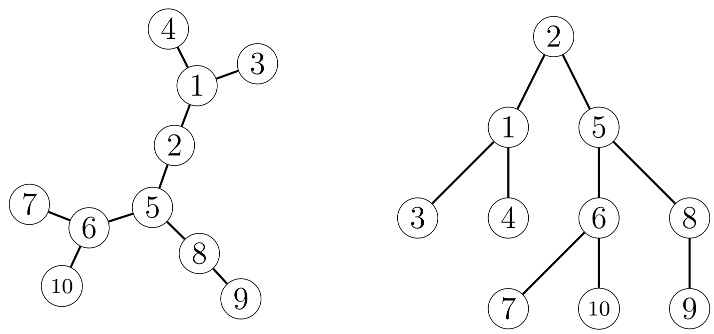

Figure 1.4: An example of a tree and a rooted tree.

Finally, we will say that a (full) spanning subgraph of G without cycles is a *(full) spanning forest* of G. A *spanning tree* is a full spanning forest of a connected G.

1.2 Building sets

In this section we recall the definition of a building set and introduce the notion of edge for these objects.

Definition 1. A *building set* $\mathcal{B}$ on $[n]$ is a collection of nonempty subsets of $[n]$ such that

(B1) If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

(B2) $\mathcal{B}$ contains the singletons $\{i\}$, for all $i \in [n]$.

For a building set $\mathcal{B}$ on $[n]$ and $I \subseteq [n]$, the *restriction* of $\mathcal{B}$ to I is the building set $\mathcal{B}|_I = \{J \in \mathcal{B} : J \subseteq I\}$. The set $\mathcal{B}_{\max} \subseteq \mathcal{B}$ consists of the inclusion-maximal subsets of $\mathcal{B}$. Their elements are pairwise disjoint subsets that partition $[n]$. The *connected components* of $\mathcal{B}$ are the restrictions $\mathcal{B}|_I$, for each $I \in \mathcal{B}_{\max}$. $\mathcal{B}$ is connected if $\mathcal{B}_{\max} = [n]$.

We note that this notion encloses the idea of connected sets, where each element of $\mathcal{B}$ represents precisely a connected subset of $[n]$.

Example 1.2.1. Consider the simple building set given by

$$\mathcal{B}_n := \{\{1\}, \ldots, \{n\}, [n]\},$$

consisting only of the singletons and the whole $[n]$. In this case, $(\mathcal{B}_n)_{\max} = \{[n]\}$, thus $\mathcal{B}_n$ is connected. Also, if $I = \{i_1, \ldots, i_m\} \subset [n]$, then $(\mathcal{B}_n)|_I = \{\{i_1\}, \ldots, \{i_m\}\}$ is not connected. Note that we can interpret n as a connected set connecting the points $1, 2, \ldots, n$. We can visualize $\mathcal{B}_n$ in Figure 1.5.

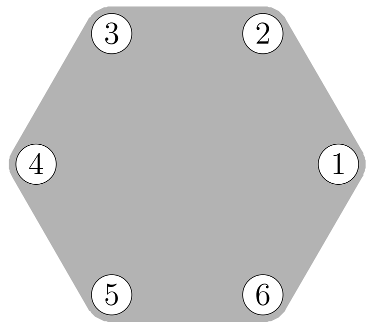

Figure 1.5: A visualization of $\mathcal{B}_n$.

Example 1.2.2 (Graphical building sets). For a graph G on $V = [n]$, the collection

$$\mathcal{B}(G) := \{\emptyset \neq I \subseteq [n] \mid G|_I \text{ is connected}\}$$

is a building set. We say that the building sets of the form $\mathcal{B}(G)$ are *graphical*. For instance, $\mathcal{B}_n$ in Example 1.2.1 is not a graphical building set.

Note that the connected components of $\mathcal{B}(G)$ correspond to the vertex sets of the connected components of G. Moreover, $\mathcal{B}(G)|_J = \mathcal{B}(G|_J)$, for any $J \subseteq [n]$.

In analogy with graphical building sets, we can introduce the notion of an edge for an arbitrary building set $\mathcal{B}$.

Definition 2. Let $\mathcal{B}$ a building set on $[n]$. An element $e \in \mathcal{B}$ is called an *edge* of $\mathcal{B}$ if $|e| \geq 2$ and e cannot be decomposed in smaller subsets $I, J \in \mathcal{B}$ such that $e = I \cup J$ and $I \cap J \neq \emptyset$. The set of edges of $\mathcal{B}$ will be denoted by $E(\mathcal{B})$.

Note that by definition the singletons are not considered as edges.

Example 1.2.3. The only edge of $\mathcal{B}_n$ is $[n]$. For the case of graphical building sets $\mathcal{B}(G)$ the edges correspond precisely to the edges of G as a graph.

Consider now the building set on $[4]$ given by

$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, [4]\}.$$

In this case $E(\mathcal{B}) = \{\{1,2\}, \{2,4\}, \{1,2,3\}\}$. In particular, $\{1,2,4\}$ is not an edge because it can be written as $\{1,2\} \cup \{2,4\}$. We can give a graphical representation of $\mathcal{B}$ by simply drawing its edges, see Figure 1.6.

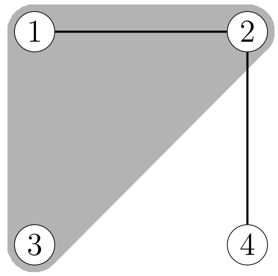

Figure 1.6: The edges of $\mathcal{B}$.

It is worth noticing that in general we may have edges $e, e' \in E(\mathcal{B})$ such that $e \subseteq e'$, as it happens in the previous example. Note that this situation does not occur in the case of graphs.

1.3 The involution principle

In this section we recall the following sieve method known as the *involution principle*. It will turn out to be very useful to simplify finite sums by pairing terms in an adequate way. For several applications of this principle the reader may consult [1].

In general, a way of pairing the elements of a finite set Ω consists of choosing a bijective map $\varphi : \Omega \to \Omega$ such that $\varphi(\varphi(x)) = x$, for all $x \in \Omega$ and such that $\mathrm{Fix}(\varphi) = \emptyset$, where $\mathrm{Fix}(\varphi) = \{y \in \Omega \mid \varphi(y) = y\}$ is the set of fixed points of φ. In fact, a pairing is a way to write

$$\Omega = \bigcup_{j=1}^{m} \{x_j, \varphi(x_j)\},$$

where these subsets are pairwise disjoint and are formed by exactly two elements. Note that the existence of a pairing impose that $|\Omega|$ is even. The map φ is usually called an *involution*.

Proposition 1.1. Let R be a commutative ring and let Ω be a finite set. Consider a map $\omega : \Omega \to R$ and the finite sum

$$S = \sum_{x \in \Omega} \omega(x).$$

Assume there is an involution $\varphi : \Omega \to \Omega$ such that

$$\omega(\varphi(x)) = -\omega(x), \qquad \text{for all } x \in \Omega.$$

Then, the following assertions hold:

1. If φ has no fixed points, then $S = 0$.

2. If φ has fixed points, then
$$S = \sum_{x \in \mathrm{Fix}(\varphi)} \omega(x).$$

Proof. 1. If φ has no fixed points, then φ is pairing the elements of Ω. Thus, $|\Omega| = 2m$, where m is the number of different pairs $\{x_1, \varphi(x_1)\}, \ldots, \{x_m, \varphi(x_m)\}$. Therefore, we can write
$$S = \sum_{j=1}^{m} \omega(x_j) + \omega(\varphi(x_j)) = 0,$$
since each sum is zero by hypothesis.

2. The map φ induces an involution $\varphi : \Omega \setminus \mathrm{Fix}(\varphi) \to \Omega \setminus \mathrm{Fix}(\varphi)$ without fixed points. Thus, the result follows from (1). □

In the situation of the previous proposition φ is usually called a *sign-reversing involution*.

1.4 The Möbius inversion formula

The goal of this section is to recall Möbius inversion formula for finite posets. Our exposition is based on [20] where the classical proofs can be consulted.

Consider a *partially ordered set* (*poset*) P, where the order is given by the binary relation $<$. Recall that $x, y \in P$ are compatible if $x \leq y$ or $y \leq x$. Otherwise, they are called incompatible. If $x < y$, we denote by
$$[x, y]_P = [x, y] := \{z \in P \mid x \leq z \leq y\}$$
the closed interval determined by x and y. In particular, $[x, x] = \{x\}$. We will denote by $\mathrm{Int}(P)$ the collection of all intervals in P. We also recall that a poset is *locally finite* if every interval $[x, y] \in \mathrm{Int}(P)$ is finite. Note also that each interval $[x, y]$ defines a poset by simply restricting the relation $<$ to its elements.

Two posets P and Q are isomorphic if there is a strictly increasing bijective map $\psi : P \to Q$, i.e., $x < y$ if and only if $\psi(p) < \psi(y)$, for all $x, y \in P$. If this is the case, each interval $[x, y]_P$ is isomorphic to $[\psi(x), \psi(y)]_Q$.

Example 1.4.1. Particular examples include $\mathbb{N}$ and $\mathbb{Z}$ with the usual order, $[n]$ with the order inherited from $\mathbb{N}$, and also $\mathbb{Z}$ with the usual division relation.

Examples of particular interest for us are the following.

Example 1.4.2 (The Boolean algebra). Let $\mathbb{B}_n$ be the poset on $\mathcal{P}([n])$ (the power set of $[n]$ consisting of the 2^n subsets of $[n]$) where $A \leq B$ in $\mathbb{B}_n$ whenever $A \subseteq B \subseteq [n]$. The poset $\mathbb{B}_n$ is called the *Boolean algebra* on $[n]$.

In the same way, we can consider the Boolean algebra $\mathbb{B}_X$, for any finite set X. It is clear that $\mathbb{B}_X$ is isomorphic to $\mathbb{B}_n$, where $n = |X|$.

Example 1.4.3 (Finite graphs). Let us denote by $\mathcal{G}$ the collection of all finite graphs. Both, the spanning subgraph and the full spanning subgraph relations give partial orders on the set $\mathcal{G}$. We will write

$$H \leq G, \qquad \text{(resp.)} \quad H \preceq G,$$

when H is a spanning subgraph (resp. full spanning subgraph) of G. For instance, $H \preceq G$ in Figure 1.2.

The poset $\mathcal{G}$ with any of the relations $\leq$ or $\preceq G$ is a locally finite poset. In this setting, note that T is a finite tree if and only if T is a minimal connected graph in $(\mathcal{G}, \preceq)$, i.e., T is connected and when we remove any $e \in E(T)$, the resulting graph T' is disconnected ($k(T') > k(T) = 1$).

On the other hand, the definition of $\mathcal{G}$ implies that for any $H \preceq G$ we have that

$$[H, G]_{\mathcal{G}} \cong [\emptyset\,,\, E(G) \setminus E(H)]_{\mathbb{B}_{E(G)}},$$

i.e., these intervals are isomorphic. This follows by noticing that a full spanning subgraph $H \preceq H' \preceq G$ is obtained from H by adding edges in $E(G) \setminus E(H)$, see Figure 1.7 for an example.

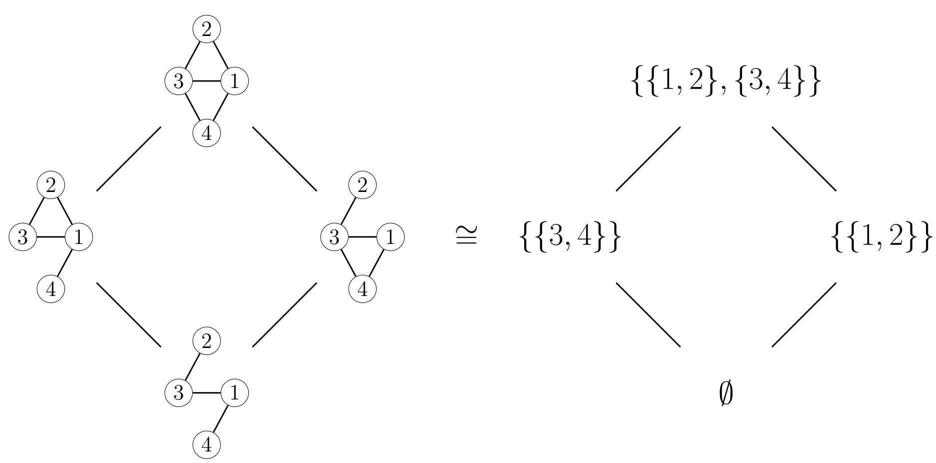

Figure 1.7: The isomorphism of intervals in $\mathcal{G}$ and a Boolean algebra.

Example 1.4.4 (Rooted trees). Every rooted tree T on $[n]$ induces a poset on $[n]$ where the order relation

$$j <_T i$$

means that j is a descendant of i in T, see Figure 1.4 for an example.

Recall that the *Möbius function* $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ can be defined recursively by the rules

$$\mu([x,y]) = \begin{cases} 1, & \text{if } x = y, \\ -\sum\limits_{x \leq z < y} \mu([x,z]), & \text{if } x < y. \end{cases}$$

This a recursive formula that for small posets can be computed directly. Moreover, isomorphic intervals have the same Möbius function. However, in rare cases it can be found by direct inspection and general techniques are required to find it.

Example 1.4.5. Consider $[n]$ with the usual order. By definition $\mu([i,i]) = 1$ and thus $\mu([i,i+1]) = -\mu([i,i]) = -1$. In the same way, $\mu([i,i+2]) = -\mu([i,i]) - \mu([i,i+1]) = 0$ and in general $\mu([i,k]) = 0$ for all $1 \leq i < i+1 < k \leq n$.

Example 1.4.6. The Möbius function in $\mathbb{B}_n$ is well-known and is given explicitly by

$$\mu([A,B]) = (-1)^{|B|-|A|},$$

for all $A \leq B$ in $\mathbb{B}_n$. Hence,

$$\mu([H,G]) = \mu([\emptyset\,,\, E(G) \setminus E(H)]) = (-1)^{|E(G)|-|E(H)|},$$

for any $H \preceq G$ in $\mathcal{G}$.

The importance of the Möbius function of a poset P is highlighted by the following inversion theorem.

Proposition 1.2 (Möbius inversion theorem). Let P be a finite poset and let $f, g : P \to K$ functions with values in a field K of characteristic zero. The following statements are equivalent:

1. $f(y) = \sum_{x \leq y} g(x)$, for all $y \in P$.
2. $g(y) = \sum_{x \leq y} \mu([x,y]) f(x)$, for all $y \in P$.

Naturally, this also holds for functions with values in rings R such as $\mathbb{Z}[t]$ or $\mathbb{Z}[x_1, \ldots, x_n]$ as they can be embedded in fields of characteristic zero (for instance, in their fields of fractions).

Example 1.4.7 (Finite differences). If we have a sequence defined in $\mathbb{R}$ by

$$g(n) = \sum_{i=1}^{n} f(i), \quad \text{then } f(n) = g(n) - g(n-1), \quad n \geq 2,$$

which is the previous inversion formula for the poset $[n]$.

Example 1.4.8 (The Principle of Inclusion-Exclusion). When $P = \mathbb{B}_n$, the Möbius inversion theorem is known as the Principle of Inclusion-Exclusion. It claims that for maps $f, g : \mathcal{P}([n]) \to K$,

$$f(B) = \sum_{A \subseteq B} g(A) \quad \text{if and only if} \quad g(B) = \sum_{A \subseteq B} (-1)^{|B|-|A|} f(A),$$

for all $B \subseteq [n]$.

Example 1.4.9. Consider maps $f, g : \mathcal{G} \to \mathbb{Z}$. Then

$$f(G) = \sum_{H \preceq G} g(H) \quad \text{if and only if} \quad g(G) = \sum_{H \preceq G} (-1)^{|E(G)|-|E(H)|} f(H),$$

for every finite graph $G \in \mathcal{G}$.

2

μ-POLYNOMIALS OF GRAPHS

The goal of this chapter is to build the μ-polynomials of graphs. As we shall see, we can do this in a recursive way. We will also compute explicitly the μ-polynomials for the infinite families of complete, path and cyclic graphs. The calculations are made directly from the recurrences that define them and some of them are based in the use of generating series. We highlight that the formulas obtained for $\mu(C_n)$ are new. In fact, we will relate $\mu(P_n)$ and $\mu(C_n)$. These are written in terms of classical Narayana polynomials, and in the former case, also in terms of the Narayana polynomials of type B. The basic definitions and recurrences in this Chapter are taken from the work of González D'León and Wachs in [11].

2.1 The construction of μ-polynomials

We start with some remarks on partitions. Let us recall that a partition π of a finite set V is a collection $\pi = \{B_1, \ldots, B_m\}$ of pairwise disjoint nonempty subsets B_j of V such that $V = \bigcup_{j=1}^{m} B_j$. In particular, $|V| = |B_1| + \cdots + |B_m|$.

Let G be a graph on $[n]$. A partition π of $[n]$ is said to be *bond* in G if for every block $B \in \pi$, the graph $G|_B$ is connected. We denote by

$$\Pi_G$$

the set of bond partitions of G. Note that if G is connected, then $[n] \in \Pi_G$.

For a bond partition π of $[n]$ we define the *restricted* graph $G|\pi$ to be the disjoint union

$$G|_\pi = \bigcup_{B \in \pi} G|_B.$$

We are ready to define the main subject of this work, namely, the μ-polynomial of a graph. This name refers to the relation with the Möbius function of the following poset of weighted bond partitions. We highlight that in [11] the authors introduced this poset as an extension of weighted partition posets.

We say that a *weighted partition* of a finite graph G is a set $\boldsymbol{\pi} = \{I_1^{\nu_1}, \ldots, I_k^{\nu_k}\}$ where

- $\pi = \{I_1, \ldots, I_k\} \in \Pi_G$ is the *underlying bond partition*.
- For all $i = 1, \ldots, k$ we have $\nu_i \in \{0, 1, \ldots, |I_i| - 1\}$.

The sets $I_i^{\nu_i}$ are called *weighted sets*. The *poset of weighted partitions* $\mathcal{W}\Pi_G$ is the set of weighted partitions of G with an order relation defined for a pair

$$\{I_1^{\nu_1}, \ldots, I_k^{\nu_k}\} \leq \{J_1^{\mu_1}, \ldots, J_l^{\mu_l}\}$$

whenever:

- $\{I_1, \ldots, I_k\} \leq \{J_1, \ldots, J_l\}$ in Π_V
- if $J_r = I_{i_1} \cup \cdots \cup I_{i_s}$ then $\mu_r - (\nu_{i_1} + \cdots + \nu_{i_s}) \in \{0, 1, ..., s-1\}$.

The *μ-polynomial* of a graph G is defined as the generating polynomial of the Möbius function of the maximal intervals of $\mathcal{W}\Pi_G$, which are indexed by an integer $j = 0, \ldots, n-1$, i.e.,

$$\mu_G(t) := \sum_{j=0}^{n-1} \mu_{\mathcal{W}\Pi_G}([\hat{0}, \{[n]^j\}])t^j.$$

Here we are using the notation $\hat{0} := \hat{0}_n = \{1|2|\ldots|n\}$ to denote the minimal bond partition consisting of all singletons. Let us give an example to fix ideas.

Example 2.1.1. Consider the poset of weighted partitions $\mathcal{W}\Pi_{P_3}$ for the path graph P_3. We can picture it in Figure 2.1 where the lines represent the order relation defined above.

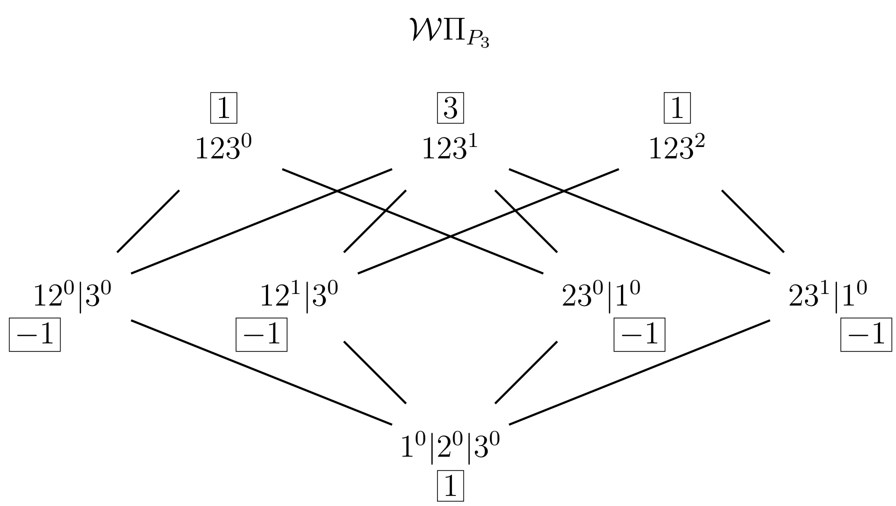

Figure 2.1: The poset $\mathcal{W}\Pi_{P_3}$.

Using the recursive definition of the Möbius function, we can compute μ and obtain the values that are indicated in the boxes of the previous figure. In particular, we find that

$$\mu_{P_3}(t) = \mu_{\mathcal{W}\Pi_{P_3}}([\hat{0}, \{[3]^0\}]) + \mu_{\mathcal{W}\Pi_{P_3}}([\hat{0}, \{[3]^1\}])t + \mu_{\mathcal{W}\Pi_{P_3}}([\hat{0}, \{[3]^2\}])t^2 = t^2 + 3t + 1.$$

In [11] is shown that the μ-polynomials of graphs can be recursively defined. This can be proved by studying the poset structure of $\mathcal{W}\Pi_G$. The following will be our starting point to work with these polynomials.

Definition/Theorem 2.1 (González D'León - Wachs [11]). The μ-polynomials of finite graphs can be computed recursively as follows:

1. We set $\mu_G(t) = 1$ when $|V| = 1$.
2. If G is a graph with connected components $G_1, \ldots, G_k$, then
$$\mu_G(t) = \mu_{G_1}(t) \cdots \mu_{G_k}(t).$$
3. If G is a connected graph with $n = |V| \geq 1$ we define
$$\mu_G(t) = -\sum_{\pi \in \Pi_G \setminus \{[n]\}} [|\pi|]_t \prod_{B \in \pi} \mu_{G|_B}(t). \tag{2.1}$$

Here we use the notation

$$[d]_t := 1 + t + \cdots + t^{d-1}, \qquad d \in \mathbb{N}^+,$$

for this symmetric polynomial which plays the role of the t-analogue to the constant d, as it converges to it as $t \to 1$.

Example 2.1.2. For the path graph P_2 the set of bond partitions is $\Pi_{P_2} = \{1|2, [2]\}$. Therefore, (2.1) shows that

$$\mu_{P_2}(t) = -(t + 1).$$

Consider now the connected graphs with three vertices P_3 and $K_3 = C_3$.

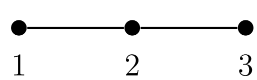

Figure 2.2: The path graph P_3.

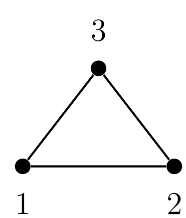

Figure 2.3: The complete graph K_3.

For P_3 its set of bond partitions is $\Pi_{P_3} = \{[3], 1|23, 3|12, 1|2|3\}$. Then (2.1) gives

$$\mu_{P_3}(t) = -2[2]_t(\mu_{P_2}(t) \cdot 1) - [3]_t(1 \cdot 1 \cdot 1) = 2(1+t)^2 - (1+t+t^2) = t^2 + 3t + 1.$$

Similarly, the bond partitions of C_3 are $\Pi_{C_3} = \{[3], 1|23, 2|13, 3|12, 1|2|3\}$. Then (2.1) gives

$$\mu_{C_3}(t) = -3[2]_t(\mu_{P_2}(t) \cdot 1) - [3]_t(1 \cdot 1 \cdot 1) = 3(1+t)^2 - (1+t+t^2) = 2t^2 + 5t + 2.$$

There are some simple properties deduced from the recurrence defining the μ-polynomials.

Proposition 2.1. The following assertions hold for a finite graph $G = (V, E)$:

1. $\mu_G(t) \in \mathbb{Z}[t]$.
2. $\mu_G(t)$ has degree at most $|V(G)| - 1$.
3. If $\mu_G(t)$ has degree exactly $|V(G)| - 1 = n - 1$ when G is connected, then $\mu_G(t)$ is symmetric (or palindromic), i.e., $\mu_G(t) = t^{\deg(\mu_G)-1}\mu_G(1/t)$. In particular, $\mu_G(-1) = 0$ for n even.

Proof. (1) is clear from the definition since $[d]_t \in \mathbb{Z}[t]$, for all $d \in \mathbb{N}^+$.
(2) By induction on n. The assertion holds for $G = (V, E)$ and $|V| = 1$ since $\mu_G(t) = 1$. If we assume it holds for all graphs with $|V| < n$, we have two cases. If G has connected components $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$, then

$$\deg(\mu_G) = \deg(\mu_{G_1}) + \cdots + \deg(\mu_{G_k}) \leq (|V_1| - 1) + \cdots + (|V_k| - 1) = |V| - k < |V| - 1,$$

so the conclusion holds in this case. Finally, if G is connected,

$$\begin{aligned}\deg(\mu_G) &\leq \max_{\pi \in \Pi_G \setminus \{[n]\}} \left\{\deg([|\pi|]_t) + \sum_{B \in \pi} \deg(\mu_{G|_B})\right\} \\ &\leq \max_{\pi \in \Pi_G \setminus \{[n]\}} \left\{|\pi| - 1 + \sum_{B \in \pi}(|B| - 1)\right\} = \max_{\pi \in \Pi_G \setminus \{[n]\}} \{|\pi| - 1 + |V| - |\pi|\} = |V| - 1,\end{aligned}$$

as required. The statement follows from the principle of induction.
(3) Again by induction on n. If the assertion holds for all connected graphs with a number of vertices less than n, and G is a connected graph on $[n]$, then

$$\begin{aligned}t^{n-1}\mu_G(1/t) &= -\sum_{\pi \in \Pi_G \setminus \{[n]\}} t^{|\pi|-1}[|\pi|]_{1/t} \prod_{B \in \pi} t^{|B|-1}\mu_{G|_B}(1/t) \\ &= -\sum_{\pi \in \Pi_G \setminus \{[n]\}} [|\pi|]_t \prod_{B \in \pi} \mu_{G|_B}(t) = \mu_G(t).\end{aligned}$$

Note we used the symmetry of the polynomials $[d]_t$: $t^{d-1}[d]_{1/t} = [d]_t$. The same holds if G has connected components $G_1, \dots, G_k$ since $\deg(\mu_G) = \sum_{j=1}^k \deg(\mu_{G_j})$ and thus

$$t^{\deg(\mu_G)}\mu_G(1/t) = \prod_{j=1}^k t^{\deg(\mu_{G_j})}\mu_{G_j}(1/t) = \prod_{j=1}^k \mu_{G_j}(t) = \mu_G(t).$$

Finally, $\mu_G(-1) = (-1)^{n-1}\mu_G(-1) = -\mu_G(-1)$, so $\mu_G(-1) = 0$ when n is even, as needed. □

Remark 2.1.1. As we shall see in Chapter 4, assertion (3) always holds, i.e., if G is connected, then $\mu_G(t)$ has degree exactly $|V(G)| - 1$. Moreover, the coefficients of μ_G have the same sign. In fact, the sign of the coefficients is $(-1)^{|V(G)|-1}$. In this way, we will write

$$|\mu|_G(t) := (-1)^{|V(G)|-1}\mu_G(t)$$

for the corresponding signless polynomial, i.e., $|\mu|_G(t)$ has non-negative coefficients.

Remark 2.1.2. There are another system of recurrences that can be used to define the μ-polynomial of a graph. In fact, we can consider the *contracted* graph $G/_\pi$ as the graph on the vertex set $V(G/_\pi) := \pi$ and edge set

$$E(G/_\pi) = \{\{B_i, B_j\} \mid B_i, B_j \in \pi \text{ are such that } \{a, b\} \in E(G) \text{ for some } a \in \pi_i, b \in \pi_j\}.$$

In Figure 2.4 we illustrate this construction for a graph on [7] and the bond partition given by $\pi = \{B_1 = \{1,4,6\}, B_2 = \{2,3,7\}, B_3 = \{5\}\}$.

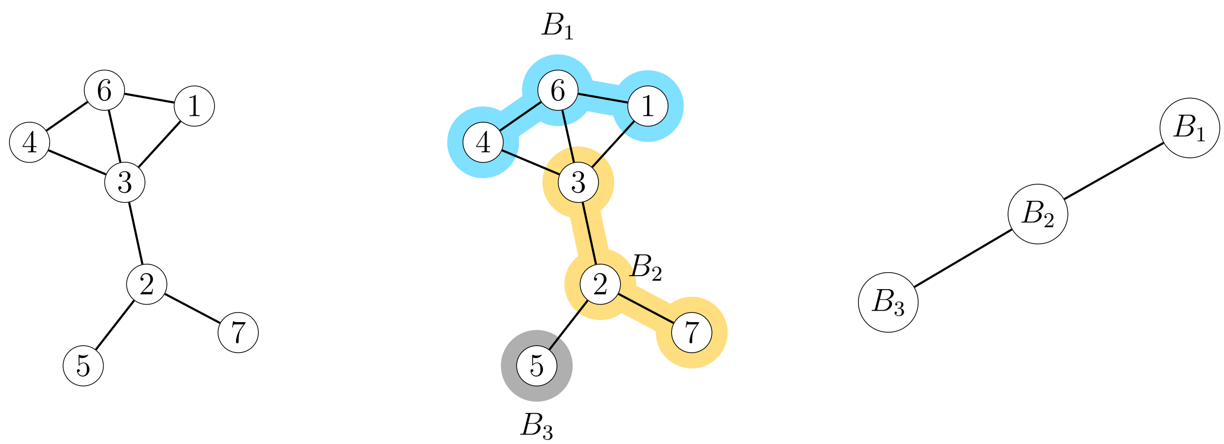

Figure 2.4: An example of a contracted graph by a bond partition.

In this way, we can replace recurrence (2.1) by the recurrence

$$\mu_G(t) = -\sum_{\pi \in \Pi_G \setminus \{\hat{0}\}} \mu_{G/\pi}(t) \prod_{B \in \pi} [|B|]_t. \tag{2.2}$$

However, we are not going to use this approach in the current work.

Remark 2.1.3. Although we are interested in the case of univariate μ-polynomials, it is worth to mention that there is a more general version of μ-polynomials in a countable family of variables $\boldsymbol{x} = (x_1, x_2, \dots)$. In fact, for the general case we consider the complete homogeneous symmetric polynomials

$$H_m(\boldsymbol{x}) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

as the formal series consisting of all monomials in $\boldsymbol{x}$ of total degree m. They can be described by the formal relation

$$\sum_{j=0}^{\infty} H_j(\boldsymbol{x}) z^n = \prod_{n=1}^{\infty} \frac{1}{1 - x_n z}.$$

In this general setting, the recurrences defining $\mu_G(\boldsymbol{x})$ associated to a graph G on $[n]$ are as follows:

1. If $n = 1$, we set $\mu_\bullet(\boldsymbol{x}) = 1$.
2. If G is a graph with connected components $G_1, \dots, G_k$, then

$$\mu_G(\boldsymbol{x}) = \mu_{G_1}(\boldsymbol{x}) \cdots \mu_{G_k}(\boldsymbol{x}).$$

3. If G is connected, then we have the following two equivalent recursions

$$\mu_G(\boldsymbol{x}) = -\sum_{\pi \in \Pi_G \setminus [n]} H_{|\pi|-1}(\boldsymbol{x}) \prod_{B \in \pi} \mu_{G|_B}(\boldsymbol{x}), \tag{2.3}$$

$$\mu_G(\boldsymbol{x}) = -\sum_{\pi \in \Pi_G \setminus \hat{0}} \mu_{G/\pi}(\boldsymbol{x}) \prod_{B \in \pi} H_{|B|-1}(\boldsymbol{x}). \tag{2.4}$$

Note that specializing at $\boldsymbol{x} = (1, t, 0, 0, \dots)$, we find the symmetric polynomials

$$[d]_t := H_{d-1}(1, t, 0, 0, \dots) = 1 + t + \cdots + t^{d-1}, \qquad d \in \mathbb{N},$$

having as ordinary generating series

$$\sum_{d=0}^{\infty} [d]_t z^d = \frac{1}{(1-z)(1-tz)}. \tag{2.5}$$

In particular, formulas (2.3) and (2.4) specialize to (2.1) and (2.2). Therefore, we recover

$$\mu_G(t) = \mu_G(1, t, 0, 0, \dots)$$

as needed.

The remaining of this chapter is devoted to find explicit formulas for the μ-polynomials associated to the graphs $K_n.P_n$ and C_n, for any $n \ge 3$. From now on, given a finite graph G we will use the notation

$$\Pi_G = \bigcup_k \Pi_{G,k},$$

where $\Pi_{G,k}$ denotes the set of bond partitions of G with exactly k blocks.

2.2 Complete graphs

In this section we compute the μ-polynomials of the complete graphs K_n. As we shall see, the calculation can be deduced from the formula of the composition of exponential generating functions using partitions and from the compositional inverse of a series involving exponential formulas.

More specifically, we will use the following proposition. The proof can be consulted in [21], Theorem 5.1.4.

Proposition 2.2 (The composition formula). Let K be a field and consider the exponential formal power series $f(t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n$ and $g(t) = 1 + \sum_{m=1}^{\infty} \frac{b_m}{m!} t^m$. Then their composition $(g \circ f)(t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$ is well-defined and its coefficients are given by $c_0 = 1$ and

$$c_n = \sum_{\pi = \{B_1, \ldots, B_k\}} a_{|B_1|} \cdots a_{|B_k|} \, b_k,$$

where the sum is taken over all partitions π of $[n]$.

With this formula at hand, we can proceed. Let K_n be the complete graph on $[n]$. For $n = 1$, $\mu_{K_1}(t) = 1$. In general, since any partition of $[n]$ is a bond partition of K_n, the recurrence (2.1) takes the form

$$\mu_{K_n}(t) + \sum_{k=2}^{n} [k]_t \sum_{\{B_1, \ldots, B_k\} \in \Pi_{K_n, k}} \mu_{K_{|B_1|}}(t) \cdots \mu_{K_{|B_k|}}(t) = 0,$$

where the inner sum is taken over all partitions of $[n]$ with k blocks. Using the previous proposition, the previous equation means that

$$\sum_{n=1}^{\infty} \frac{[n]_t}{n!} z^n = \frac{e^{tz} - e^{z}}{t-1}, \text{ and } K(z,t) = \sum_{n=1}^{\infty} \frac{\mu_{K_n}(t)}{n!} z^n$$

are compositional inverses one of each other, i.e.,

$$\frac{e^{tK(z,t)} - e^{K(z,t)}}{t-1} = z.$$

To solve this equation we recall the formula

$$\left(\frac{e^{\alpha x} - e^{\beta x}}{(\alpha - \beta) e^{\alpha x} e^{\beta x}} \right)^{\circ(-1)} = \sum_{n=1}^{\infty} \prod_{j=1}^{n-1} ((n-j)\alpha + j\beta) \frac{x^n}{n!}$$

for parameters $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$ given in [9, Example 1.7.2]. We refer the reader to Appendix A.1 for a direct proof of this fact.

Choosing $\alpha = -1, \beta = -t$ leads to $K(z,t)$. In this way, we obtain the following result.

Proposition 2.3. The μ-polynomials of the complete graph K_n is given by

$$\mu_{K_n}(t) = (-1)^{n-1} \prod_{j=1}^{n-1} (jt + n - j). \tag{2.6}$$

In this case, we can check the validity of Remark 2.1.1 since all coefficients have the same sign, the polynomial has degree $n-1$, and it is symmetric. Note that the leading coefficient as well as the constant term of $|\mu|_{K_n}(t)$ is precisely

$$|\mu|_{K_n}(0) = (n-1)!.$$

The first few values of these polynomials are

$$\begin{aligned}
|\mu|_{K_2}(t) &= t + 1,\\
|\mu|_{K_3}(t) &= 2t^2 + 5t + 2,\\
|\mu|_{K_4}(t) &= 6t^3 + 26t^2 + 26t + 6,\\
|\mu|_{K_5}(t) &= 24t^4 + 154t^3 + 269t^2 + 154t + 24,\\
|\mu|_{K_6}(t) &= 120t^5 + 1044t^4 + 2724t^3 + 2724t^2 + 1044t + 120,\\
|\mu|_{K_7}(t) &= 720t^6 + 8028t^5 + 28636t^4 + 42881t^3 + 28636t^2 + 8028t + 720.
\end{aligned}$$

Another property that is evident from the general formula (2.6) is that μ_{K_n} has only real roots and they are simple.

2.3 Path graphs

We move now to the case of the path graphs P_n. In this case, the calculation depends on the usual formula for the composition of formal power series. We will see that in this case the μ-polynomials are a well-known family of polynomials known as the Narayana polynomials.

Proposition 2.4. Let K be a field and consider the formal power series $f(t) = \sum_{n=1}^{\infty} a_n t^n$ and $g(t) = b_0 + \sum_{m=1}^{\infty} b_m t^m$. Then their composition $(g \circ f)(t) = \sum_{n=0}^{\infty} c_n t^n$ is well-defined and its coefficients are given by $c_0 = b_0$ and

$$c_n = b_1 a_n + \sum_{j=2}^{n} b_k \sum_{k_1+\cdots+k_j=n} a_{k_1} \cdots a_{k_j},$$

where the sum is taken over all possible solutions $(k_1, \ldots, k_j) \in (\mathbb{N}^+)^j$ in positive integers of $k_1 + \cdots + k_j = n$.

We start by identifying the bond partitions associated to the path graph P_n on $[n]$. In fact, having a bond partition π in P_n corresponds to choose $1 \le j \le n$ and a solution

$(k_1, \ldots, k_j)$ in positive integers of $k_1 + \cdots + k_j = n$, being $j = |\pi|$. Indeed, any other partition π having a block B containing non-consecutive terms cannot be bond.

Some examples of this correspondence are $\pi = (1|2|\ldots|n)$, where $j = n$ and $(1, 1, \ldots, 1)$. Also, for $\pi = [n]$ we have $j = 1$ and $(k_1 = n)$.

Therefore, the recurrence (2.1) defining $\mu_{P_n}(t)$ takes the form $\mu_{P_n}(t) = 1$ and

$$\mu_{P_n} + \sum_{j=2}^{n} [j]_t \sum_{k_1+\cdots+k_j=n} \mu_{P_{k_1}} \ldots \mu_{P_{k_j}} = 0, \qquad n \geq 2.$$

This means that the series

$$\mathcal{H}(z,t) = \sum_{j=1}^{\infty} [j]_t z^j = \frac{z}{(1-z)(1-tz)}, \qquad \mathcal{L}(z,t) = \sum_{j=1}^{\infty} \mu_{P_j}(t) z^j$$

(recall (2.5)) are compositional inverses one of each other. Inverting $\mathcal{H}$ as a power series in z, i.e., solving $\frac{\zeta}{(1-\zeta)(1-t\zeta)} = z$ for ζ, or equivalently, solving the quadratic equation

$$(zt)\zeta^2 - (1 + z(1+t))\zeta + z = 0,$$

we find that

$$\mathcal{L}(z,t) = \frac{1 + z(1+t) - \sqrt{1 + 2z(1+t) + z^2(t-1)^2}}{2zt}. \tag{2.7}$$

Despite its appearance, the previous generating series gives rise to explicit formulas in terms of the *Narayana polynomials* and *Narayana numbers*, having several applications in Combinatorics. The reader may consult [14] for more information.

The Narayana polynomials $\mathcal{N}_n(t)$ are defined by means of the recurrence $\mathcal{N}_1(t) = 1$ and

$$\mathcal{N}_n(t) = (1+t)\mathcal{N}_{n-1}(t) + t \sum_{j=1}^{n-2} \mathcal{N}_j(t)\mathcal{N}_{n-1-j}(t), \qquad n \geq 2.$$

In terms of the generating series

$$\mathcal{N}(z,t) := \sum_{n=1}^{\infty} \mathcal{N}_n(t) z^n$$

this means that

$$\begin{aligned}
\mathcal{N}(z,t) &= z + \sum_{n=2}^{\infty} \mathcal{N}_n(t) z^n \\
&= z + \sum_{n=2}^{\infty} \left[(1+t)\mathcal{N}_{n-1}(t) + t \sum_{j=1}^{n-2} \mathcal{N}_j(t)\mathcal{N}_{n-1-j}(t) \right] z^n \\
&= z + (1+t)z\mathcal{N}(z,t) + tz\mathcal{N}(z,t)^2,
\end{aligned}$$

or equivalently, that

$$(zt)\mathcal{N}(z,t)^2 + (z(1+t)-1)\mathcal{N} + z = 0.$$

Solving this equation we see that

$$\mathcal{N}(z,t) = \frac{1 - z(1+t) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2tz}. \tag{2.8}$$

In particular, we see that

$$\mathcal{L}(z,t) = -\mathcal{N}(-z,t).$$

This proves the following proposition.

Proposition 2.5. The μ-polynomials of the path graphs P_n are given by

$$\mu_{P_n}(t) = (-1)^{n-1}\mathcal{N}_n(t).$$

Once again, we see that the coefficients of $\mu_{P_n}(t)$ have the same sign and that the polynomials are symmetrical. In terms of the generating series of the Narayana polynomials, this is equivalent to say that

$$\frac{1}{z}\mathcal{N}\left(tz, \frac{1}{t}\right) = \mathcal{N}(z,t),$$

which readily follows from (2.8).

The explicit coefficients of the Narayana polynomials

$$\mathcal{N}_n(t) = \sum_{k=0}^{n-1} N_{n,k} t^k$$

are well-known and are given by the Narayana numbers $N_{n,k}$ which are non-negative integers. In fact, it is known that

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k}\binom{n}{k+1}, \qquad 0 \le k < n.$$

In particular,

$$N_{n,0} = N_{n,n-1} = 1$$

and $\mu_{P_n}(t)$ has degree exactly $n-1$.

For further use, we remark the alternative formula

$$\mathcal{N}_n(t) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} t^{k-1}, \qquad n \ge 1, \tag{2.9}$$

that follows by recalling that $\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k}$.

The first few values of the $\mu(P_n)$- and Narayana polynomials are

$$
\begin{aligned}
|\mu|_{P_2}(t) &= t+1,\\
|\mu|_{P_3}(t) &= t^2+3t+1,\\
|\mu|_{P_4}(t) &= t^3+6t^2+6t+1,\\
|\mu|_{P_5}(t) &= t^4+10t^3+20t^2+10t+1,\\
|\mu|_{P_6}(t) &= t^5+15t^4+50t^3+50t^2+15t+1,\\
|\mu|_{P_7}(t) &= t^6+21t^5+105t^4+175t^3+105t^2+21t+1.
\end{aligned}
$$

2.4 Cyclic graphs

This final section deals with the explicit calculation of the μ-polynomials of the cyclic graphs C_n and their generating series. We will see that they can be written in terms of the $\mu(P_n)$- and the Narayana polynomials of type B. We highlight that the results of this section are new.

To describe the recurrence (2.1) for $G = C_n$, it is necessary to describe all bond partitions of this graph. These are close related to the bond partitions of P_n. In fact, note that given $\gamma \in \Pi_{P_n}$ and $i \in [n]$, we obtain a bond partition of C_n rolling up clock-wise P_n on C_n, matching the vertex 1 of P_n with the vertex i of C_n, see Figure 2.5.

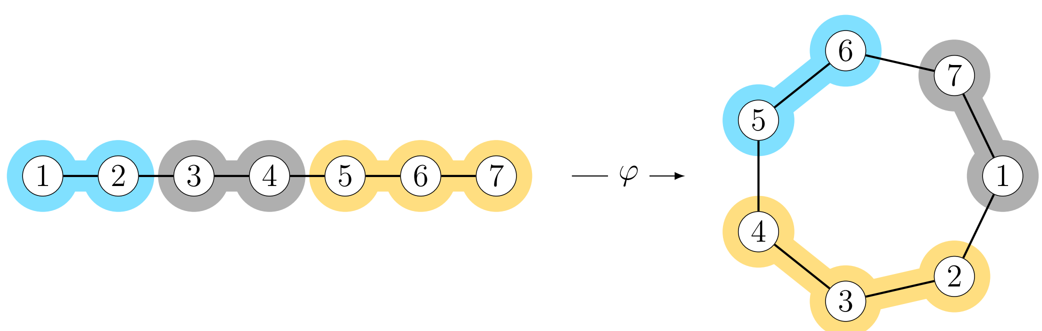

Figure 2.5: A bond partition of P_n gives bond partitions on C_n.

More precisely, if we set

$$g_i : [n] \longrightarrow [n], \qquad x \longmapsto x+i-1 (\text{mod } n),$$

we obtain the map

$$F : \Pi_{P_n} \times [n] \to \Pi_{C_n}, \qquad F(\gamma, i) = \{g_i(A) \,:\, A \in \gamma\}.$$

Since g_i is a permutation of $[n]$ (given by a translation), F is well-defined. Note that F preserves the number of blocks, i.e., $F(\gamma, i) \in \Pi_{C_n,k}$ if and only if $\gamma \in \Pi_{P_n,k}$.

Note that F is onto, but not injective. In fact, given $\pi \in \Pi_{C_n}$, if we select a block $B \in \pi$ and let $i = \min B$, there is a unique $\gamma \in \Pi_{P_n}$ such that $F(\gamma, i) = \pi$. However, different choices of blocks produce different γ.

To identify this correspondence, for each level $\Pi_{P_n,k} \times [n]$, we can unfold by blocks and obtain the desired bijection. To this end, given $\pi \in \Pi_{C_n}$, consider the map $f_\pi : \pi \longrightarrow [n]$, $B \longmapsto \min B$.

Lemma 2.6. If $k > 1$ and $C^*_{n,k} = \{(\pi, B) : \pi \in \Pi_{C_n,k}, B \in \pi\}$, the map defined by

$$\varphi_k : \Pi_{P_n,k} \times [n] \longrightarrow C^*_{n,k} \qquad (\gamma, i) \longmapsto (\pi, B') = (F(\gamma, i), f_\pi^{-1}(i)),$$

is bijective.

Proof. By the definition of F, $B' = f_\pi^{-1}(i)$ is the block of $\pi = F(\gamma, i)$ having i as least element, and $g_i^{-1}(B')$ is the block of γ containing 1. Therefore, φ_k consists of simply marking the block of π that contains i.

Given $\pi \in \Pi_{C_n,k}$ and $B \in \pi$, if we set $i = f_\pi(B)$ and $\gamma = \{g_i^{-1}(B) : B \in \pi\}$, then $\varphi_k(\gamma, i) = (\pi, B)$. Thus, φ_k is onto. The previous discussion shows γ is unique, so φ_k is also injective. □

Remark 2.4.1. This lemma gives a combinatorial proof of the equality

$$k|\Pi_{C_n,k}| = n|\Pi_{P_n,k}|. \tag{2.10}$$

In fact, simply note that $|C^*_{n,k}| = k|\Pi_{C_n,k}|$. We can give another proof as follows. For P_n we have that $|\Pi_{P_n,k}| = \binom{n-1}{k-1}$, since this is the number of solutions in positive integers of $j_1 + \cdots + j_k = n$, see Section 2.3. For C_n, partition this into $k > 1$ connected components corresponds to remove k edges from C_n. Therefore, $|\Pi_{C_n,k}| = \binom{n}{k}$ from which (2.10) follows.

We are in position to write (2.1) explicitly. First, grouping the sum for bond partitions with the same number of blocks, we have

$$\mu_{C_n}(t) = -\sum_{k=2}^{n} [k]_t \sum_{\pi \in \Pi_{C_n,k}} \prod_{B \in \pi} \mu_{C_n|_B}(t), \qquad n \geq 2.$$

Notice that $\mu_{C_n|_B}(t) = \mu_{P_{|B|}}(t)$, for each block $B \in \pi$. In particular, if $\pi = F(\gamma, i)$, for some $i \in [n]$, then $\prod_{B\in\pi} \mu_{P_{|B|}}(t) = \prod_{A\in\gamma} \mu_{P_{|A|}}(t)$ since F preserves the size of each block. Then Lemma 2.6 shows that

$$\begin{aligned} k \sum_{\pi \in \Pi_{C_n,k}} \prod_{B \in \pi} \mu_{P_{|B|}}(t) &= \sum_{(\pi,B') \in C^*_{n,k}} \prod_{B \in \pi} \mu_{P_{|B|}}(t) \\ &= \sum_{(\gamma,i) \in \Pi_{P_n,k} \times [n]} \prod_{A \in \gamma} \mu_{P_{|A|}}(t) = n \sum_{\gamma \in \Pi_{P_n,k}} \prod_{A \in \gamma} \mu_{P_{|A|}}(t). \end{aligned}$$

In conclusion, we find the recurrence

$$\mu_{C_n}(t) = -\sum_{k=2}^{n} \frac{n}{k}[k]_t \sum_{\pi \in \Pi_{P_n,k}} \prod_{B\in\pi} \mu_{P_{|B|}}(t), \qquad n \geq 2.$$

This means that the generating series

$$\mathcal{C}(z,t) := \sum_{n=1}^{\infty} \mu_{C_n}(t) z^n$$

satisfies

$$\begin{aligned}
\mathcal{C}(z,t) &= z - \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n} \frac{[k]_t}{k} \sum_{\pi\in\Pi_{P_n,k}} \prod_{B\in\pi} \mu_{P_{|B|}}(t) \right] n z^n + \sum_{n=2}^{\infty} n\mu_{P_n}(t) z^n \\
&= z - \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{[k]_t}{k} \sum_{\pi\in\Pi_{P_n,k}} \prod_{B\in\pi} \mu_{P_{|B|}}(t) \right] n z^n + \sum_{n=1}^{\infty} n\mu_{P_n}(t) z^n \\
&= z - z\frac{\partial \mathcal{F}}{\partial z}(z,t) + z\frac{\partial \mathcal{L}}{\partial z}(z,t)
\end{aligned}$$

where $\mathcal{L}$ is as in (2.7) and

$$\mathcal{F}(z,t) = \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{[k]_t}{k} \sum_{\pi\in\Pi_{P_n,k}} \prod_{B\in\pi} \mu_{P_{|B|}}(t) \right] z^n = \left(\sum_{n=1}^{\infty} \frac{[n]_t}{n} z^n \right) \circ \mathcal{L}(z,t).$$

Noticing that $\partial_z \left(\sum_{n=1}^{\infty} [n]_t z^n / n \right) = \mathcal{H}(z,t)/z$, an application of the chain rule gives

$$z\frac{\partial \mathcal{F}}{\partial z} = z\frac{\mathcal{H}(\mathcal{L},t)}{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial z} = \frac{z^2}{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial z} = \frac{z}{\sqrt{1 + 2z(1+t) + z^2(1-t)^2}}.$$

From here, a straightforward calculation proves the following result.

Proposition 2.7. The generating series of the $\mu(C_n)$-polynomials is

$$\mathcal{C}(z,t) = z + \frac{1 - 2tz^2 + z(t+1) - \sqrt{1 + 2z(1+t) + z^2(1-t)^2}}{2tz\sqrt{1 + 2z(1+t) + z^2(1-t)^2}}.$$

To obtain a closed formula for $\mu_{C_n}(t)$, we will use of the *Narayana polynomials of type B* [8] defined as

$$W_n(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

If $n = 0$, $W_0(t) = 1$. We collect some of their properties in the following lemma.

LEMMA 2.8. The Narayana polynomials of type B satisfy the recurrence

$$W_n(t) = (t+1)W_{n-1}(t) + 2(n-1)t\mathcal{N}_{n-1}(t), \tag{2.11}$$

where $\mathcal{N}_n$ is nth Narayana polynomial. Moreover, their generating series is

$$W(z,t) = \sum_{n=0}^{\infty} W_n(t)z^n = \frac{1}{\sqrt{1-2z(t+1)+z^2(1-t)^2}}. \tag{2.12}$$

Proof. Equation (2.11) follows expanding the square of $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and recalling (2.9). In fact,

$$\begin{aligned}
W_n(t) &= \sum_{k=0}^{n} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]^2 t^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k}^2 t^k + 2\sum_{k=0}^{n} \binom{n-1}{k}\binom{n-1}{k-1} t^k + \sum_{k=0}^{n} \binom{n-1}{k-1}^2 t^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k}^2 t^k + 2(n-1)t \sum_{k=1}^{n-1} \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1} t^{k-1} + t\sum_{k=0}^{n-1} \binom{n-1}{k}^2 t^k \\
&= (t+1)W_{n-1}(t) + 2(n-1)t\mathcal{N}_{n-1}(t).
\end{aligned}$$

Note that (2.11) is equivalent to say that $W(z,t)$ satisfies

$$W(z,t) = 1 + z(1+t)W(z,t) + 2tz^2\frac{\partial \mathcal{N}}{\partial z}.$$

Solving for $W(z,t)$, and recalling from (2.8) that

$$\frac{\partial \mathcal{N}}{\partial z} = \frac{1 - z(1+t) - \sqrt{1-2z(t+1)+(1-t)^2}}{2tz^2\sqrt{1-2z(t+1)+(1-t)^2}}, \tag{2.13}$$

a brief calculation leads to (2.12).

□

We are in position to prove an explicit formula for $\mu_{C_n}(t)$ in terms of the Narayana polynomials $\mathcal{N}_n(t)$ and $W_n(t)$.

THEOREM 2.9. The μ-polynomial $\mu_{C_n}(t)$ associated to the cyclic graph C_n on $[n]$ is given by the formula

$$\begin{aligned}
\mu_{C_n}(t) &= (-1)^{n-1}\left[n\mathcal{N}_n(t) - W_{n-1}(t)\right] \\
&= (-1)^{n-1}\sum_{k=0}^{n-1}\left[\frac{n}{k+1}\binom{n}{k}\binom{n-1}{k} - \binom{n-1}{k}^2\right] t^k.
\end{aligned} \tag{2.14}$$

Proof. If $\mathcal{P}(z,t) = z + \sum_{n=2}^{\infty} [n\mathcal{N}_n(t) - W_{n-1}(t)]\, z^n$, (2.12) and (2.13) show that

$$\begin{aligned}\mathcal{P}(z,t) &= z + z\frac{\partial \mathcal{N}}{\partial z} - zW(z,t) \\ &= z + \frac{1 - z(t+1) - 2tz^2 - \sqrt{1 - 2z(t+1) + z^2(1-t)^2}}{2tz\sqrt{1 - 2z(t+1) + z^2(1-t)^2}}.\end{aligned}$$

Since $\mathcal{C}(z,t) = -\mathcal{P}(-z,t)$, the result follows from Proposition 2.7. □

The first cyclic polynomials are given by

$$\begin{aligned}|\mu|_{C_2}(t) &= t + 1, \\ |\mu|_{C_3}(t) &= 2t^2 + 5t + 2, \\ |\mu|_{C_4}(t) &= 3t^3 + 15t^2 + 15t + 3, \\ |\mu|_{C_5}(t) &= 4t^4 + 34t^3 + 64t^2 + 34t + 4, \\ |\mu|_{C_6}(t) &= 5t^5 + 65t^4 + 200t^3 + 200t^2 + 65t + 5, \\ |\mu|_{C_7}(t) &= 6t^6 + 111t^5 + 510t^4 + 825t^3 + 510t^2 + 111t + 6.\end{aligned}$$

In this case, it follows from the general formulas that

$$|\mu|_{C_n}(0) = n - 1,$$

and that $|\mu|_{C_n}(t)$ has positive coefficients. The last assertion is a consequence of the inequality $\frac{n}{k+1}\binom{n}{k} > \binom{n-1}{k}$, holding for $k = 0, 1, \ldots, n-1$. Finally, μ_{C_n} is symmetric since $\mathcal{N}_n(t)$ and $W_{n-1}(t)$ are symmetric. Once again, μ_{C_n} has degree exactly $n - 1$.

3

Relations between μ- and h-polynomials

The aim of this chapter is to recall the h-polynomials of building sets $\mathcal{B}$ and their representation in terms of $\mathcal{B}$-forests. Inspired in the case of cyclic graphs, we will prove that it is possible to compute the $\mu(G)$-polynomial in terms of h-polynomials of graphical building sets $\mathcal{B}(H)$, where H is a full-spanning subgraphs of G. Moreover, the formula we will obtain can be inverted by means of Möbius inversion theorem, thus also allowing us to find h-polynomials of graphical building sets in terms of μ-polynomials of its full spanning subgraphs.

3.1 h-polynomials and $\mathcal{B}$-forests

In this section we recall the h-polynomial $h_{\mathcal{B}}(t)$ of a building set $\mathcal{B}$ and its expansion in terms of $\mathcal{B}$-forests. These results will be crucial in the development of the chapter since they will allow us to prove the first main result of this work.

The recurrence definition of the h-polynomials is as follows and it is due to A. Postnikov.

Definition/Theorem 3.1 (c.f. Theorem 7.11 [15]). The h-polynomials associated to building sets is determined by the following recurrence relations:

1. If $\mathcal{B}$ consists of a singleton, $h_{\mathcal{B}}(t) = 1$.

2. If $\mathcal{B}$ has connected components $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, then

$$h_{\mathcal{B}}(t) = h_{\mathcal{B}_1}(t) h_{\mathcal{B}_2}(t) \cdots h_{\mathcal{B}_k}(t).$$

3. If $\mathcal{B}$ is connected, then

$$h_{\mathcal{B}}(t) = \sum_{I \subsetneq [n]} (t-1)^{n-|I|-1} h_{\mathcal{B}|_I}(t).$$

Example 3.1.1. Recall the building set $\mathcal{B}_n$ from Example 1.2.1. Since $(\mathcal{B}_n)|_I$ is a disconnected building set formed by $|I|$ singletons, $h_{(\mathcal{B}_n)|_I} = 1$, for all $I \subsetneq [n]$. Thus, the previous recurrence gives

$$h_{\mathcal{B}_n}(t) = \sum_{I \subsetneq [n]} (t-1)^{n-|I|-1} = \sum_{j=0}^{n-1} \binom{n}{j} (t-1)^{n-1-j} = \frac{t^n - 1}{t-1} = 1 + t + \cdots + t^{n-1}.$$

The h-polynomials admit an expansion in terms of $\mathcal{B}$-forests, which we recall now.

Definition 3 ($\mathcal{B}$-forest). A rooted forest F on $[n]$ is a $\mathcal{B}$-*forest* for a building set $\mathcal{B}$ on $[n]$ when

(BF1) For any $i \in [n]$, we have $F_{\leq i} \in \mathcal{B}$.

(BF2) For $k \geq 2$ incomparable nodes $i_1, \ldots, i_k \in [n]$ of F, one has $\bigcup_{j=1}^{k} F_{\leq i_j} \notin \mathcal{B}$.

(BF3) The sets $F_{\leq i}$, for all roots i of F, are exactly the maximal elements of the building set $\mathcal{B}$.

The connected components of $\mathcal{B}$-forest are called $\mathcal{B}$-*trees*. Thus, the connected components of a $\mathcal{B}$-forest are the $\mathcal{B}$-tress associated to the connected components of $\mathcal{B}$ as a building set.

Remark 3.1.1. The previous definition is designed for general building sets. However, for graphical ones, we can assume that $k = 2$ in (BF2) when G is connected. In fact, let T be a $\mathcal{B}(G)$-tree and assume that for any pair nodes $i, j \in [n]$, we have that $T_{\leq i} \cup T_{\leq j} \notin \mathcal{B}(G)$. By contradiction, assume there are nodes $i_1, \ldots, i_k$ of T such that $T_{\leq i_1} \cup \cdots \cup T_{\leq i_k} \in \mathcal{B}(G)$, where $k \geq 2$. Since each $T_{\leq i_j}$ gives a connected subgraph of G, and their union also gives a connected subgraph of G, there must be a couple of indexes $a, b \in \{i_1, \ldots, i_k\}$ such that $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(G)$, which is not possible. Thus, (BF2) holds for any $k \geq 2$.

We will be using an alternative description of a $\mathcal{B}$-tree presented originally in [16].

Proposition 3.1 (c.f. Proposition 8.5 [16]). Let $\mathcal{B}$ be a connected building set on $[n]$ and $\mathcal{B}_1, \ldots, \mathcal{B}_r$ the connected components of the restricted building set $\mathcal{B}|_{[n]\setminus\{i\}}$. All the $\mathcal{B}$-trees T with root $i \in [n]$ are obtained recursively by connecting to i the roots of $\mathcal{B}_j$-trees T_j for $j = 1, \ldots, r$.

Varying i in Proposition 3.1 we obtain all possible $\mathcal{B}$-trees. If $\mathcal{B}$ is not connected, we apply this process to each of its connected components in order to construct all possible $\mathcal{B}$-forests.

Example 3.1.2. Consider the graph G on $[8]$ displayed in Figure 3.1. The $\mathcal{B}$-tree T in the picture is obtained as follows. First, choose 4 as root of T. The graph G' obtained from G by removing 4 is still connected, so 4 has only one child in T. Now choose 5 as the next vertex in G'. By doing so, G' gives two graphs: a path one on $\{6, 7, 8\}$ and a kite one on $\{1, 2, 3, 4\}$. Thus 5 has two children in T. By continuing this way we arrive at T as in the figure.

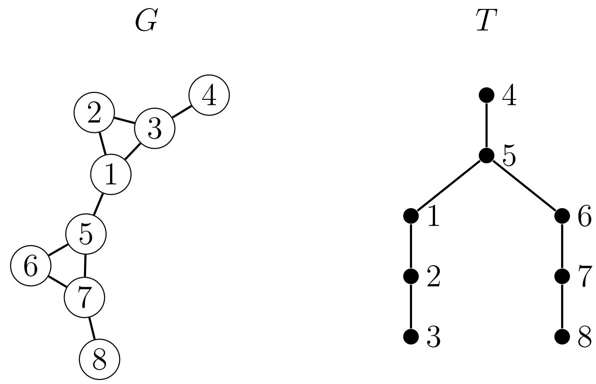

Figure 3.1: T is an example of a $\mathcal{B}(G)$-tree.

In a $\mathcal{B}$-forest F, an edge $\{v, w\} \in E(F)$ is a *descent* if $w <_T v$ and $w < v$, according to the usual order of $\mathbb{N}$. We denote the total numbers of descents in F as $\mathrm{des}(F)$. For instance, $\mathrm{des}(T) = 1$ in the tree of Figure 3.1 and the only descent is at the edge $\{1, 5\}$.

In these terms, we have the following fundamental result from [16].

Proposition 3.2 (Corollary 8.4[16]). If $\mathcal{B}$ is a building set on $[n]$, then

$$h_{\mathcal{B}}(t) = \sum_{F \text{ a } \mathcal{B}\text{-forest}} t^{\mathrm{des}(F)},$$

where the sum is taken over all possible $\mathcal{B}$-forests.

Example 3.1.3. For the building set $\mathcal{B}_n$ of Example 1.2.1, since $(\mathcal{B}_n)|_{[n]\setminus\{j\}}$ has $n-1$ connected components, the only $\mathcal{B}_n$-tree T_j with root j has $n-1$ children, and thus $\mathrm{des}(T_j) = j - 1$, see Figure 3.2. The previous result proves again that

$$h_{\mathcal{B}_n}(t) = \sum_{j=1}^{n} t^{j-1},$$

as expected.

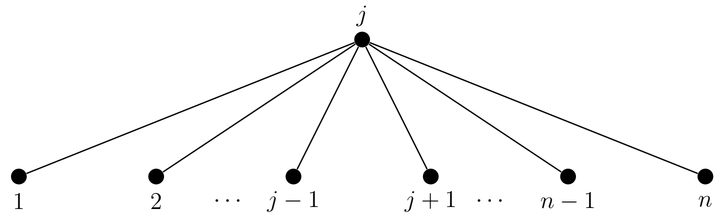

Figure 3.2: The $\mathcal{B}_n$-tree T_j.

3.2 μ-polynomials in terms of h-polynomials

It is known that the h-polynomials of the graphical building sets for the path and cyclic graphs are precisely the Narayana polynomials

$$h_{\mathcal{B}(P_n)}(t) = \mathcal{N}_n(t), \qquad h_{\mathcal{B}(C_n)}(t) = W_{n-1}(t), \tag{3.1}$$

respectively, [16, Section 10]. In these terms, (2.14) takes the form

$$\mu_{C_n}(t) = (-1)^{n-1} n h_{\mathcal{B}(P_n)}(t) + (-1)^n h_{\mathcal{B}(C_n)}(t). \tag{3.2}$$

We can explain this formula in terms of spanning subgraphs. Recalling Example 1.1.5, we see that $\mu_{C_n}(t)$ coincides with the sum over all its spanning graphs H of $(-1)^{|E(H)|} h_{\mathcal{B}(H)}(t)$. The relation (3.2) is not an isolated fact. Indeed, it inspired the following theorem expressing $\mu_G(t)$ in terms of h-polynomials of graphical building sets.

Theorem 3.3.

$$\mu_G(t) = \sum_{H \preceq G} (-1)^{|E(H)|} h_H(t), \tag{3.3}$$

where the sum is taken over all full spanning subgraphs H of G.

In particular, if G is a connected tree on $[n]$, its only spanning subgraph is itself and $|E(G)| = n - 1$. Therefore, we obtain the following result proving Conjecture 1.

Corollary 3.4. If G is a tree on $[n]$, then

$$\mu_G(t) = (-1)^{n-1} h_G(t).$$

Example 3.2.1. Theorem 3.3 also gives a new proof of the values in (3.1). In fact, by Proposition 2.5 and the previous corollary, we see that

$$h_{P_n}(t) = (-1)^{n-1} \mu_{P_n}(t) = \mathcal{N}_n(t).$$

Moreover, the previous theorem proves that (3.2) holds. Thus, we can solve this equation for $h_{\mathcal{B}(C_n)}(t)$ and recover the value $h_{\mathcal{B}(C_n)}(t) = W_{n-1}(t)$ from Theorem 2.9.

The remaining of this section is devoted to the proof of Theorem 3.3. We start with the following lemma.

Lemma 3.5. The following statements are equivalent:

1. For any $G \in \mathcal{G}$
$$\mu_G(t) = \sum_{H \preceq G} (-1)^{|E(H)|} h_H(t), \tag{3.4}$$
where the sum is over all full spanning subgraphs H of G.

2. For any connected $G \in \mathcal{G}$ with $|V(G)| > 1$ we have that
$$\sum_{H \leq G} (-1)^{|E(H)|} [k(H)]_t h_{\mathcal{B}(H)}(t) = 0, \tag{3.5}$$
where the sum is over all spanning subgraphs H of G.

Proof. Since $\mu_G(t) = (-1)^0 h_G(t) = 1$ when $G = (\bullet, \emptyset)$, the idea of the proof will be to show that the right-hand side of (3.4) satisfies the recursion (2.1) when G is a graph on vertex set with $|V| > 1$. For such graph then define
$$Q_G(t) := \sum_{H \preceq G} (-1)^{|E(H)|} h_H(t).$$

Note that if G has connected components $G_1, \ldots, G_k$, then
$$Q_G(t) = Q_{G_1}(t) \cdots Q_{G_k}(t).$$

Indeed, a full spanning subgraph $H \preceq G$ is the disjoint union of $H_1, \ldots, H_k$, where each $H_j \preceq G_j$. The formula follows since $|E(H)| = |E(H_1)| + \cdots + |E(H_k)|$ and $h_H = h_{H_1} \cdots h_{H_k}$, thanks to Theorem 3.1.

Therefore, it is sufficient to prove that $Q_G(t)$ satisfies the recurrence of equation (2.1) when G is connected. For this we want to prove, in equivalent form, that
$$\sum_{\pi \in \Pi_G} [|\pi|]_t \prod_{B \in \pi} Q_{G|_B}(t) = 0.$$

Recalling that for each bond partition $\pi \in \Pi_G$, the connected components of $G|_\pi$ are of the form $G|_B$, where $B \in \pi$. The previous equation is equivalent to
$$\sum_{\pi \in \Pi_G} \sum_{H \preceq G|_\pi} (-1)^{|E(H)|} [|\pi|]_t h_H(t) = 0. \tag{3.6}$$

Note that the two sums in equation (3.6) can replaced by a sum indexed on the set
$$I_1 := \{(\pi, H) \mid \pi \in \Pi_G, H \preceq G|_\pi\}.$$

Let

$$I_2 := \{H \mid H \le G\}.$$

We show that there is a simple bijection between the sets I_1 and I_2 which allow us to rewrite (3.6) in a simpler form.

Indeed, if $(\pi, H) \in I_1$, then H can be regarded as a subgraph of G since it is a full spanning subgraph $H \preceq G|_\pi$, and hence satisfies $V(H) = V(G|_\pi) = V(G)$. Conversely, let $H \in I_2$ with connected components $H_1, \ldots, H_k$. The set $\pi_H = \{V(H_1), \ldots, V(H_k)\}$ is a bond partition of G since connectivity in H implies connectivity in G and $\cup_{j=1}^{k} V(H_j) = V(H) = V(G)$ (see Figure 3.3 for an example). Then $(\pi_H, H) \in I_1$ given that each H_j is connected. Finally, it is clear that the maps $(\pi, H) \in I_1 \to H \in I_2$ and $H \in I_2 \to (\pi_H, H) \in I_1$ are inverses of each other.

Since $k(H) = k(G|_\pi) = |\pi|$ for $(\pi, H) \in I_1$, we conclude that the double sum in (3.6) can be written as in equation (3.5). □

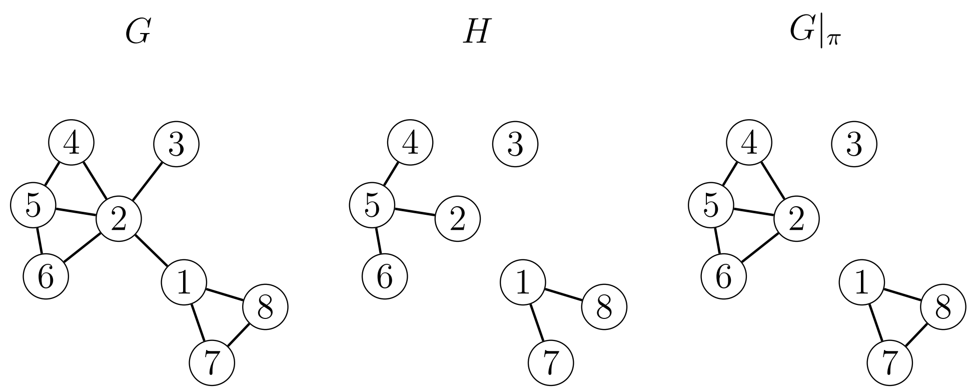

Figure 3.3: The pair (π_H, H) for the given H, where $\pi_H = \{178|2456|3\}$.

A sign-reversing involution. Due to Lemma 3.5, in order to prove Theorem 3.3 we now have to show that equation (3.5) holds for any connected $G \in \mathcal{G}$ with $|V(G)| > 1$.

The idea to prove equation (3.5) is the following. We write first (3.5) in an equivalent form as a sum over a suitable set of combinatorial objects Ω_G together with a signed weight-function

$$\omega : \Omega_G \to \mathbb{Z}[t]$$

where $\omega(x)$ is the contribution to the left-hand side of equation (3.5) of $x \in \Omega_G$. Then, we want to construct a sign-reversing involution

$$\varphi : \Omega_G \to \Omega_G,$$

i.e., one satisfying $\omega(\varphi(x)) = -\omega(x)$. If the involution φ is free of fixed-points we would have concluded that the left-hand side of equation (3.5) vanishes by invoking the Involution Principle explained in Section 1.3.

Based on Proposition 3.2, we can replace $h_{\mathcal{B}(H)}(t)$ by the sum of $t^{\text{des}(F)}$ over all $\mathcal{B}(H)$-forest F. Moreover, if F is a $\mathcal{B}(H)$-forest with $H \leq G$, we can sort its roots $\tau_0 < \tau_1 < \cdots < \tau_{k(H)-1}$ in ascending order, according to the natural order of $\mathbb{N}$ and define $\sigma_F(\tau_i) := i$. Let

$$\Omega_G := \{(H, F, \tau) \mid H \leq G,\ F \text{ a } \mathcal{B}(H)\text{-forest, and } \tau \text{ a root of } F\},$$
$$\omega : \Omega_G \to \mathbb{Z}[t], \qquad \omega(H, F, \tau) = (-1)^{|E(H)|} t^{\operatorname{des} F + \sigma_F(\tau)}.$$

Since $[k(H)]_t = 1 + t + \cdots + t^{k(H)-1}$, equation (3.5) can be rewritten as

$$\sum_{(H,F,\tau)\in\Omega_G} \omega(H, F, \tau) = 0, \tag{3.7}$$

which is the sum for which we will build the sign-reversion involution.

Two auxiliary constructions. To proceed we require the following two auxiliary constructions from the set of $\mathcal{B}(H)$-forests F with $H \leq G$ to the set of $\mathcal{B}(H')$-forests F with $H' \leq G$ where H' is obtained by adding or removing an edge $\{v_0, v_1\} \in E(G)$ from H. We consider the following situations:

If $\{v_0, v_1\} \in E(H)$ and $v_1 \in F_{\leq v_0}$, we let $E(H') = E(H) \setminus \{\{v_0, v_1\}\}$ and define the rooted forest $F \downarrow^{v_0}_{v_1}$ by cases:

(a) if $\{v_0, v_1\}$ disconnects H, $F \downarrow^{v_0}_{v_1}$ is the forest resulting from F by removing the edge $\{v_0, a\} \in E(F)$, where a is child of v_0 such that $v_1 \in F_{\leq a}$, see Figure 3.4. Then, $F \downarrow^{v_0}_{v_1}$ has one more tree than F, having a as root.

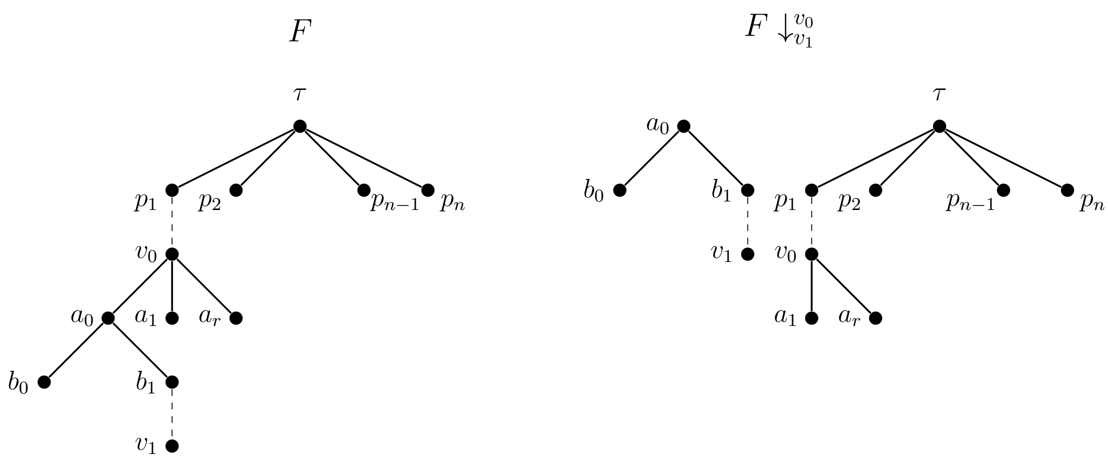

Figure 3.4: Construction of $F \downarrow^{v_0}_{v_1}$ when $\{v_0, v_1\}$ disconnect H.

(b) If $\{v_0, v_1\}$ does not disconnect H, $F \downarrow^{v_0}_{v_1} = F$.

If $\{v_0, v_1\} \notin E(H)$, we let $E(H') = E(H) \cup \{\{v_0, v_1\}\}$ and we denote by $F \uparrow^{v_1}_{v_0}$ the following rooted forest:

(a) if v_0 and v_1 are in different connected components of H, $F \uparrow_{v_0}^{v_1}$ is the forest resulting from joining v_0 to $F_{\leq a}$ by the new edge $\{v_0, a\}$, where a is the root of the tree of F containing v_1.

(b) If v_0 and v_1 are in the same connected component of H, then $F \uparrow_{v_0}^{v_1} = F$.

The proof of the next lemma is based on Proposition 3.1 describing the recursive construction of $\mathcal{B}$-forests.

Lemma 3.6. For $(H, F, \tau) \in \Omega_G$ the following assertions hold.

1. If $v \in F_{\leq \tau}$ is such that $\{\tau, v\} \in E(H)$, then $F \downarrow_v^\tau$ is a $\mathcal{B}(H')$-forest, where $H' = (V(H), E(H)\backslash\{\{\tau, v\}\})$.
2. If $\{\tau, v\} \in E(G) \setminus E(H)$ and a is the root of F such that $v \in F_{\leq a}$, then $F \uparrow_\tau^v$ is a $\mathcal{B}(H')$-forest, where $H' = (V(H), E(H) \cup \{\{\tau, v\}\})$.

Proof. We only prove (1). The proof of (2) follows a similar line of reasoning and it is omitted. Let $H \leq G$ with connected componentes $H_1, \ldots, H_r$ and assume without loss of generality that $\tau, v \in H_1$. According to the construction of $F \downarrow_v^\tau$, we have two cases:

(a.) When $\{\tau, v\}$ disconnects H, the graph H' has an additional connected component compared to H. Let us say that the component H_1 of H splits into two connected components H'_0 and H'_1 of H'. Assume without loss of generality that $\tau \in H'_0$ and $v \in H'_1$. Note that $H|_{[n]\backslash\{\tau\}}$ and $H'|_{[n]\backslash\{\tau\}}$ have the same connected components, namely, $H'_1, H_2, \ldots, H_r$ and the connected components of $H'_0|_{[n]\backslash\{\tau\}}$.

Now, let F is a $\mathcal{B}(H)$-forest formed by $\mathcal{B}(H_j)$-tress T_j for $j = 1, \ldots, r$. T_1 has $\tau \in H_1$ as root which has as many children as components of $H'_0|_{[n]\backslash\{\tau\}}$, plus the children a for which $v \in (T_1)_{\leq a}$. Therefore, $F \downarrow_v^\tau$ is formed by the trees $T_2, \ldots, T_r$, $(T_1)_{\leq a}$, and a tree T'_1 resulting from T_1 by removing $\{\tau, a\}$, which is a $\mathcal{B}(H')$-forest. Indeed, T_j is a $\mathcal{B}(H_j)$-tree for $j = 2, \ldots, r$, $(T_1)_{\leq a}$ is a $\mathcal{B}(H_0)$-tree with root a, and T'_1 is a $\mathcal{B}(H'_0)$-tree with root τ. The last holds because the subtrees attached to τ in T'_1 are constructed recursively as $\mathcal{B}$-trees from the components of $H_1|_{[n]\backslash\{\tau\}}$ that are different to H'_1, that is, from the components of $H'_0|_{[n]\backslash\{\tau\}}$.

(b.) When $\{v_0, v_1\}$ does not disconnect H, we have that $F \downarrow_{v_1}^{v_0} = F$. In this case, H and H' have the same number of connected components. Then $H|_{[n]\backslash\{\tau\}}$ and $H'|_{[n]\backslash\{\tau\}}$ have the same connected components and it follows that F is also a $\mathcal{B}(H')$-forest.

□

Consider now the map $\varphi : \Omega_G \longrightarrow \Omega_G$ defined by

$$\varphi(H, F, \tau) = \begin{cases} (H', F \downarrow_v^\tau, \tau), & \text{if } \{\tau, v\} \in E(H), \\ (H', F \uparrow_\tau^v, \tau), & \text{if } \{\tau, v\} \notin E(H), \end{cases}$$

where $v = \min\{u \in \mathbb{N} \mid \{\tau, u\} \in E(G)\}$ and H' is the subgraph of G such that $E(H') = E(H)\backslash\{\{\tau, v\}\}$ or $E(H') = E(H) \cup \{\{\tau, v\}\}$ respectively. The map φ is well-defined thanks to Lemma 3.6.

LEMMA 3.7. The map φ is a sign-reversing involution on Ω_G that is free of fixed points.

Proof. We want to check that $\varphi \circ \varphi = \mathrm{id}_{\Omega_G}$. We write $(H', F', \tau) := \varphi(H, F, \tau)$ and let $(H'', F'', \tau) := \varphi(H', F', \tau)$.

Note that the constructions (1)(a) and (2)(a), as well as (1)(b) and (2)(b), are inverses of each other. In symbols, we have that

$$(F \downarrow^{\tau}_{v}) \uparrow^{\tau}_{v} = F, \qquad (F \uparrow^{\tau}_{v}) \downarrow^{\tau}_{v} = F.$$

In the case $\{\tau, v\} \in E(H)$. By definition, $\{\tau, v\} \notin E(H')$. Thus, $E(H'') = E(H') \cup \{\{\tau, v\}\} = E(H)$ and hence $H'' = H$. Also $F'' = (F \downarrow^{\tau}_{v}) \uparrow^{v}_{\tau} = F$. Similarly, in the case where $\{\tau, v\} \notin E(H)$, we have $H'' = H$ and $F'' = F$. Therefore, checking that φ is an involution.

To prove that φ is sign reversing we need to show that

$$\omega(H, F, \tau) = (-1)^{E(H)} t^{\operatorname{des} F + \sigma_F(\tau)} = -(-1)^{E(H')} t^{\operatorname{des} F' + \sigma_{F'}(\tau)} = -\omega(H, F', \tau).$$

By construction $|E(H)| = |E(H')| \pm 1$, so $(-1)^{|E(H)|} = -(-1)^{|E(H')|}$. Thus, we are left to prove that

$$\operatorname{des} F + \sigma_F(\tau) = \operatorname{des} F' + \sigma_{F'}(\tau). \tag{3.8}$$

When $F' = F$ the equality is clear. Consider the situation when $\{\tau, v\}$ disconnects H. Let a be the child of τ such that $v \in F_{\leq a}$, which is now a root of $F' = F \downarrow^{\tau}_{v}$. We have two cases. The first case is when $\tau > a$ (as numbers in $\mathbb{N}$). In this case we have $\operatorname{des}(F') = \operatorname{des}(F) - 1$ and $\sigma_{F'}(\tau) = \sigma_F(\tau) + 1$. The second case is when $\tau < a$. In this case we have $\operatorname{des}(F') = \operatorname{des}(F)$ and $\sigma_F(\tau) = \sigma_{F'}(\tau)$. In any of these cases equation (3.8) holds. Now, in the situation when τ and v are in different connected components of H, and $\{\tau, v\}$ decrease the number of connected components with respect H, we have that $F' = F \uparrow^{v}_{\tau}$, and the same line of reasoning leads to verify equation (3.8).

Finally, the fact that φ has no fixed points follows from the fact that, in any situation, $H \neq H'$. We conclude that φ satisfies the required properties. □

We have now all the ingredients for the proof of Theorem 3.3.

Proof of Theorem 3.3. Lemma 3.7 proves equation (3.7) and hence, by the comments above, confirms the equivalent equation (3.5). The proof of Theorem 3.3 now follows from Lemma 3.5. □

3.3 From μ-polynomials to h-polynomials: a Möbius inversion formula

The Möbius inversion formula explained in Section 1.4 allows to invert the formula obtained in Theorem 3.3. In fact, we have the following alternative to compute h-polynomials of graphical building sets from μ-polynomial.

Theorem 3.8.

$$h_{\mathcal{B}(G)}(t) = \sum_{H \preceq G} (-1)^{|E(H)|} \mu_H(t),$$

where the sum is taken over all full spanning subgraphs H of G.

Proof. This is a consequence of Theorem 3.3 and the Möbius inversion theorem applied to the poset $\mathcal{G}$ of finite graphs with the full-spanning order relation. In fact, Example 1.4.9 proves that

$$\begin{aligned} (-1)^{|E(G)|} h_{\mathcal{B}(G)}(t) &= \sum_{H \preceq G} \mu([H, G]) \mu_H(t) \\ &= \sum_{H \preceq G} (-1)^{|E(G)| - |E(H)|} \mu_H(t), \end{aligned}$$

which is equivalent to the formula we want to prove. □

To conclude we remark that the previous theorem also allows to conclude the calculations of Example 3.2.1.

4

μ-TREES

The goal of this chapter is to construct a new family of forests associated to μ-polynomials in the spirit of the $\mathcal{B}$-forests associated to the h-polynomials of building sets. We will develop their main properties, including different characterizations for them as well as an analogous result to Proposition 3.2. As an application, we provide a closed formula, in terms of Narayana polynomials, for the μ-polynomials of a family of kite-like graphs.

4.1 The notion of μ-trees and μ-forests

Definition 4 (μ-forest). Let G be a graph on $[n]$. A rooted forest F on $[n]$ is an *μ-forest* for G (or simply a $\mu(G)$-forest) if

(MF1) For any $j \in [n]$, and for all children i of j, we have that $F_{\leq i} \cup \{j\} \in \mathcal{B}(G)$.

(MF2) The sets $F_{\leq i}$, for all roots i of F, are exactly the maximal elements of $\mathcal{B}(G)$.

If G is connected, we refer to such F as an *μ-tree*. Thus, the connected components of an μ-forest are the μ-trees associated to the connected components of G.

Remark 4.1.1. If T be a μ-tree for a connected graph G, then T always satisfies (BF1). In fact, if $v \in [n]$,

$$T_{\leq v} = \bigcup_{a \lessdot v} \left(T_{\leq a} \cup \{v\}\right),$$

where the union is taken over the children a of v. Since $T_{\leq a} \cup \{v\} \in \mathcal{B}(G)$, and they all have $\{v\}$ in common, then $T_{\leq v} \in \mathcal{B}(G)$, as desired.

Fix a connected graph G on $[n]$ and $v \in [n]$. We will refer to a bond partition π of $G|_{[n]\setminus\{v\}}$ as v-*admissible* if for every block $B \in \pi$, there is at least a $u \in B$ such that $\{v, u\} \in E(G)$. With this terminology, we can give an algorithmic construction for all μ-trees in analogy with Proposition 3.1 for $\mathcal{B}$-trees.

PROPOSITION 4.1. Let G be a connected graph on $[n]$ and $\pi = \{B_1, \dots, B_r\}$ a v-admissible partition of the restricted graph $G|_{[n]\setminus\{v\}}$. All the $\mu(G)$-trees T with root $v \in [n]$ are obtained recursively by connecting to v the roots of $\mu(G|_{B_j})$-trees T_j for $j = 1, \dots, r$.

Proof. The condition on admissible partitions is clearly justified by condition (MF1) in the previous definition: in order to have $T_{\leq a} \cup \{v\} \in \mathcal{B}(G)$, we require that $\mathcal{B}(G)$ restricted to $[n] \setminus \{v\}$ is a connected building set, i.e., there is an edge in G connecting $\{v\}$ with a vertex of $G|_{T_{\leq a}}$. □

In this way, we obtain the following proposition.

PROPOSITION 4.2 (ALGORITHMIC CONSTRUCTION OF μ-TREES). Every $\mu(G)$-tree of a connected graph G on $[n]$ is obtained as follows:

1. Choose a vertex v of G as the root of T.

2. Use an v-admissible partition of $G|_{[n]\setminus\{v\}}$ to split this subgraph into connected components $G_1, \dots, G_r$. Then, apply the same procedure to each G_j, and connect the selected roots $v_j \in [n] \setminus \{v\}$ to v.

3. Repeat the process a finite number of times, until exhausting $[n]$.

If G is not connected, we apply the previous process to each of its connected components in order to construct all $\mu(G)$-forests.

We conclude this section with some examples.

EXAMPLE 4.1.1. Consider the path graph P_2. Its μ-trees are obtained, first, selecting a root v. If $v = 1$, $H = (P_2)|_{[2]\setminus 1}$ has only one vertex, so the only 1-admissible partition is $\pi = \{\{2\}\}$. For $v = 2$ the reasoning is the same. Thus, there are two $\mu(P_2)$-trees Y_1 and Y_2 which are displayed in Figure 4.1.

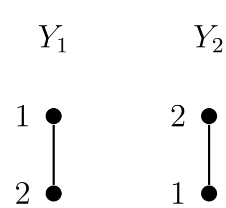

Figure 4.1: The $\mu(P_2)$-trees.

Now consider the cycle graph C_3. For this graph, if we select $v = 1$, then $H_1 = C_3|_{[3]\setminus 1} = P_2$ is the path subgraph on $\{2, 3\}$. The 1-admissible partitions are $\pi_1 = \{\{2, 3\}\}$ and $\pi_2 = \{\{2\}, \{3\}\}$. In fact, π_1 is valid since $\{1, 2\} \in E(C_3)$, and π_2 is valid since $\{1, 3\}, \{1, 2\} \in E(C_3)$. Therefore, there are three $\mu(C_3)$-trees rooted at 1. For $\mu(C_3)$-trees rooted at 2 or 3 the reasoning is the same. Figure 4.2 shows all possible $\mu(C_3)$-trees.

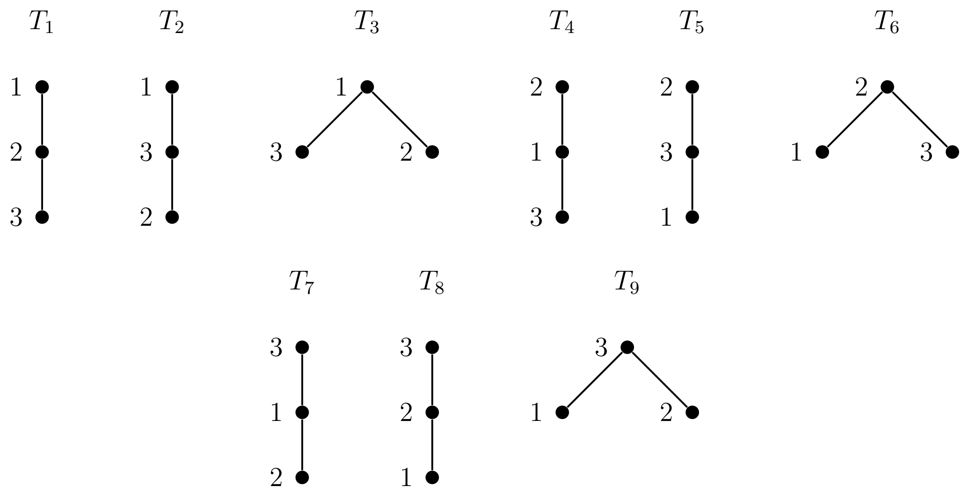

Figure 4.2: The $\mu(C_3)$-trees.

4.2 Some characterizations of μ-forests

We turn now to characterize μ-forests in terms of full spanning graphs and $\mathcal{B}$-forests.

Lemma 4.3. Let G be a graph on $[n]$. A rooted tree F on $[n]$ is a $\mu(G)$-forest if and only if there is a full spanning graph $H \preceq G$ such that F is a $\mathcal{B}(H)$-forest.

Proof. If F is a $\mathcal{B}(G)$-forest for some $H \preceq G$, (MF1) holds for F thanks to (BF1) and (BF2) in Definition 3. Therefore, F is a $\mu(G)$-forest.

Conversely, note first that by working with connected components, it is sufficient to assume that G is connected. Then, let T be a $\mu(G)$-tree. Remark 4.1.1 shows that T always satisfies (BF1) for $\mathcal{B}(G)$. We have two cases. If T satisfies (BF2), then we take $H = G$ and we are done. Otherwise, by Remark 3.1.1, there are $a, b \in [n]$ such that $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(G)$. Therefore, there are $a_1 \in T_{\leq a}$ and $b_1 \in T_{\leq b}$ with $\{a_1, b_1\} \in E(G)$. Let $v \in [n]$ be such that $a_1, b_1 \in T_{\leq v}$. By (MF1) there is a path in G from a_1 to v not containing b_1, and another path in G from b_1 to v not containing a_1. Thus, G has a cycle containing $\{a_1, b_1\}$. Let H_1 the subgraph of G such that $E(H_1) = E(G) \setminus \{a_1, b_1\}$. By construction $H_1 \preceq G$ and T is a $\mu(H_1)$-tree. In particular, T satisfies (BF1) for $\mathcal{B}(H_1)$.

At this stage, we check if T is a $\mathcal{B}(H_1)$-tree. If this is the case, we set $H = H_1$ and we are done. Otherwise, we can repeat the argument above a finite number of times until reaching a $H_r \preceq G$ for which one on the following statements hold:

1. T is a $\mathcal{B}(H_r)$-tree.
2. H_r has no cycles, i.e., it is a spanning tree of G. If T was not a $\mathcal{B}(H_r)$-tree, the previous paragraph shows that H_r must have a cycle, which is impossible. Therefore. T must be a $\mathcal{B}(H_r)$-tree.

In both cases, we finish by setting $H = H_r$. □

The previous lemma can be improved to generate a spanning tree for which a given $\mu(G)$-tree is a $\mathcal{B}$-tree. Before proceeding we introduce the following notation.

REMARK 4.2.1. Given a graph G on $[n]$, $v \in [n]$, and $A \subseteq [n]$, assume that the set $\{u' \in A : \{u', v\} \in E(G)\}$ is nonempty. In this case, we denote by

$$u_{G,v,A} = \min_{\mathbb{N}}\{u' \in A : \{u', v\} \in E(G)\},$$

as such minimum with respect to the usual order of $\mathbb{N}$.

PROPOSITION 4.4. Let G be a connected graph on $[n]$ and let T be a $\mu(G)$-tree. Then, there is a full spanning tree $\Psi = \Psi(G, T)$ of G such that T is a $\mathcal{B}(\Psi)$-tree.

Proof. For each $v \in [n]$, consider the edges

$$E_{T,v}(G) = \{\{v, u_a\} \in E(G) : u_a = u_{G,v,T_{\leq a}}, a \text{ is a child of } v\}.$$

Now, let $\Psi = \Psi(G, T)$ be the graph on $[n]$ such that

$$E(\Psi) = \bigcup_{v \in [n]} E_{T,v}(G).$$

In other words, to find Ψ, for each $v \in [n]$, check the branches $T_{\leq a}$ of T obtained by removing v, and choose the smallest $u_a \in [n]$ in each of them (in the usual order of $\mathbb{N}$) such that $\{v, u_a\}$ is an edge of G. Note we can do this starting with the vertices a at the bottom of T, then for the vertices v such that $a \lessdot v$, and progressively until reaching the root of T, see Figure 4.3 for an example.

We prove that $\Psi(G, T)$ is a full spanning tree of G by induction on n. If $n = 1$, the result is clear, so assume the proposition holds for graphs on $[k]$, for all $k < n$.

Given G and T as in the statement, let $\tau \in [n]$ be the root of T. If a is a child of τ in T, the tree $T_{\leq a}$ is a $\mu(G|_{T_{\leq a}})$-tree. By induction hypothesis, the graph $\Psi_a = \Psi(G|_{T_{\leq a}}, T_{\leq a})$ is a full spanning tree of $G|_{T_{\leq a}}$ and $T_{\leq a}$ is a $\mathcal{B}(G|_{T_{\leq a}})$-tree. By construction, if $\Psi = \Psi(G, T)$, then $\Psi|_{T_{\leq a}} = \Psi_a$. Since T is a $\mu(G)$-tree, $T_{\leq a} \cup \{\tau\} \in \mathcal{B}(G)$ and therefore $w_a = u_{G,\tau,T_{\leq a}}$ exists. Noticing that

$$E(\Psi) = \bigcup_{a \lessdot v} (E(\Psi_a) \cup \{w_a, \tau\}),$$

and that $T_{\leq a_1} \cap T_{\leq a_2} = \emptyset$ for any children $a_1 \neq a_2$ of τ, we see that τ is part of no cycle in Ψ. Thus, Ψ has no cycles, i.e., Ψ is a full spanning tree of G and T is $\mathcal{B}(\Psi)$-tree as it was required. The principle of induction allows to conclude the proof. □

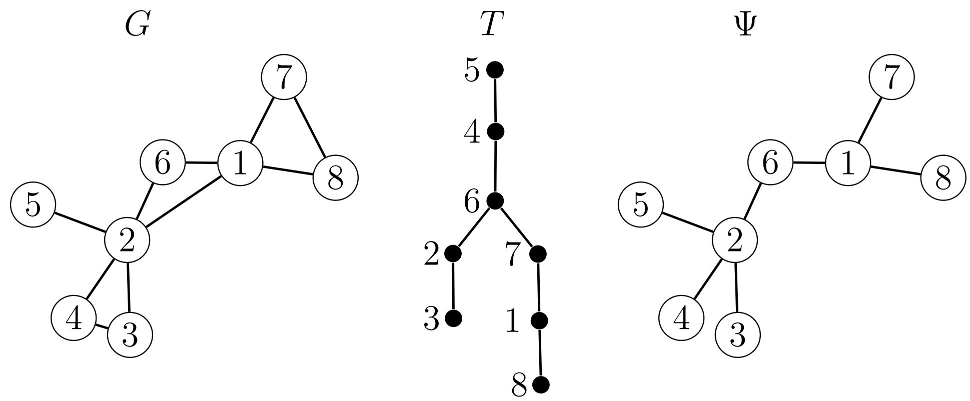

Figure 4.3: The full spanning tree $\Psi(G, T)$, for the $\mu(G)$-tree T.

In summary, we have obtained the following characterizations of μ-trees.

Corollary 4.5. Let G be a connected graph on $[n]$ and T a rooted tree on $[n]$. The following assertions are equivalent:

1. T is a $\mu(G)$-tree.
2. T is a $\mathcal{B}(H)$-tree, for some $H \preceq G$.
3. T is a $\mathcal{B}(\Psi)$-tree, for some full spanning tree Ψ of G.

4.3 μ-polynomials and μ-forests

The aim of this section is to prove Theorem 4.9 which is the analogous result to Theorem 3.2 for μ-polynomials and μ-forests. As an important consequence we will prove that the coefficients of a given μ_G polynomial have the same sign and we will determine its exact degree.

The idea is the same as in the previous chapter and the goal is to construct a suitable sign-reversing involution on the set

$$\Lambda_G := \{(H, T) : H \preceq G \text{ and } T \text{ is a } \mathcal{B}(H)\text{-tree}\},$$

for a given a connected graph G on $[n]$. In this case, the fixed points are gonna be indexed by the $\mu(G)$-trees. Our first step is the following key lemma.

Lemma 4.6. Let G be a connected graph on $[n]$ and $(H, T) \in \Lambda_G$. Let $v, u \in [n]$, and let C be a cycle in G such that $\{v, u\} \in E(C)$, where $u \in V(C) \subseteq T_{\leq v}$. Then, the following assertions are valid:

1. If C is a cycle in H, then T is a $\mathcal{B}(H')$-tree, where $E(H') = E(H) \setminus \{\{v, u\}\}$.

2. If $\{v, u\} \notin E(H)$, but C is formed in by adding $\{v, u\}$ to H, then T is a $\mathcal{B}(H'')$-tree, where $E(H'') = E(H) \cup \{\{v, u\}\}$.

Proof. (1) Since we are removing an edge from H, it is clear that (BF2) still holds. To prove that (BF1) is valid, let $q \in [n]$. We consider the following possibilities:

- If $v \in T_{\leq q}$, $H|_{T_{\leq q}}$ is a connected subgraph of H because $T_{\leq q} \in \mathcal{B}(H)$. Since $V(C) \subseteq T_{\leq v} \subseteq T_{\leq q}$, then C is a cycle in $H|_{T_{\leq q}}$. Therefore, the graph $H'|_{T_{\leq q}}$ remains connected since $E(H'|_{T_{\leq q}}) = E(H|_{T_{\leq q}}) \setminus \{\{v, u\}\}$, i.e., $T_{\leq q} \in \mathcal{B}(H')$.
- If v and q are not comparable in T, $\{v, u\} \notin E(H|_{T_{\leq q}})$, so $H'|_{T_{\leq q}} = H|_{T_{\leq q}}$ is connected.
- If $q \in T_{\leq v} \setminus \{v\}$, then $v \notin T_{\leq q}$. Thus, $\{v, u\} \notin E(H|_{T_{\leq q}})$ and so $H'|_{T_{\leq q}} = H|_{T_{\leq q}}$ is connected.

In any case, we see that $T_{\leq q} \in \mathcal{B}(H')$. Therefore, T is a $\mathcal{B}(H')$-tree as required.

(2) First, (BF1) holds since $\mathcal{B}(H) \subseteq \mathcal{B}(H'')$. Second, if (BF2) is not valid, by Remark 3.1.1 there would be $a, b \in [n]$ non-comparable in T such that $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(H'')$. Since $T_{\leq a} \cup T_{\leq b} \notin \mathcal{B}(H)$ and H and H'' only differ by an edge, then $v, u \in T_{\leq a} \cup T_{\leq b}$. Being $T_{\leq a}$ and $T_{\leq b}$ disjoint, we can assume $v \in T_{\leq a}$ and $u \in T_{\leq b}$. This contradicts that $u \in T_{\leq v}$, so (BF2) must hold. In conclusion, T is a $\mathcal{B}(H'')$-tree as we wanted to show. □

We define the map

$$\lambda : \Lambda_G \to \Lambda_G$$

taking into account the following construction, for a given $(H, T) \in \Lambda_G$. For every cycle C in G, let $v_C \in V(C)$ be such that the remaining vertices of C are below v_C in T, i.e., $V(C) \subseteq T_{\leq v_C}$. Then, let

$$u_C := u_{G, v_C, V(C)} \in T_{\leq v_C},$$

where the notation is as in Remark 4.2.1, and consider the ordered pair (v_C, u_C). Define $W_{H,T}$ as the set of all pairs (v_C, u_C) satisfying one of the following two conditions:

(i) $\{v_C, u_C\} \in E(H)$ and the cycle C is also a cycle in H.

(ii) $\{v_C, u_C\} \notin E(H)$, but C is a cycle in $(V(H), E(H) \cup \{\{v_C, u_C\}\})$.

If $W_{H,T}$ is nonempty, we set

$$(v, u)_{H,T} = (v, u) := \min_{\text{lex}} W_{H,T}$$

as the minimum of $W_{H,T}$ with respect to the lexicographical order, and the order in each component is in $\mathbb{N}$.

Using this notation, if $\{v, u\} \in E(H)$, we let H^- be the subgraph of G such that $E(H^-) = E(H)\backslash\{\{v, u\}\}$. If $\{v, u\} \notin E(H)$ we consider H^+ instead, where $E(H^+) = E(H) \cup \{\{v, u\}\}$. In these terms, we finally define

$$\lambda(H, T) := \begin{cases} (H^-, T), & \text{if } \{v, u\} \in E(H), \\ (H^+, T), & \text{if } \{v, u\} \notin E(H), \\ (H, T), & \text{if there is no } \{v, u\}. \end{cases}$$

Proposition 4.6 proves that λ maps Λ_G to itself. We illustrate this map below.

EXAMPLE 4.3.1. Consider G and $(H, T) \in \Lambda_G$ displayed in Figure 4.4.

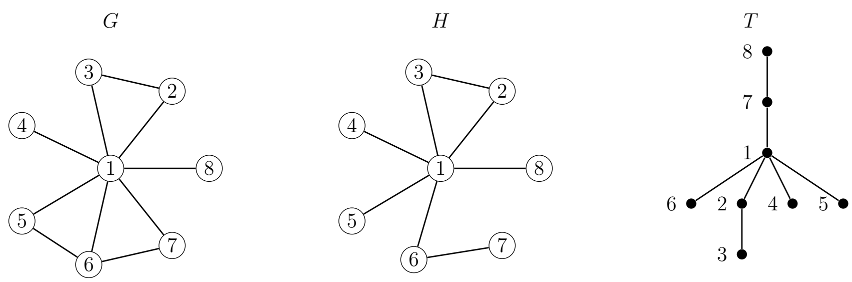

Figure 4.4: A given $H \preceq G$ and a $\mathcal{B}(H)$-tree T.

To find $\lambda(H, T)$ we compute (v_C, u_C) for the cycles in G:

- For C formed by $1, 2$ and 3, $v_C = 1$, $u_C = u_{G,1,\{1,2,3\}} = 2$, and $(v_C, u_C) = (1, 2)$. C is valid since it is a cycle in H with $\{1, 2\} \in E(H)$. C can be seen in Figure 4.5.

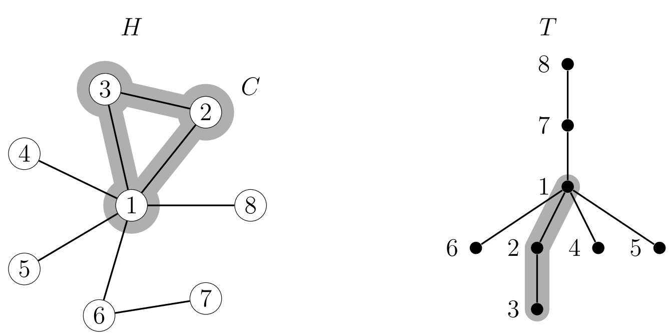

Figure 4.5: The cycle C in H and T.

- For C' formed by $1, 5$ and 6, $v_{C'} = 1$, $u_{C'} = u_{G,1,\{1,5,6\}} = 5$, and $(v_{C'}, u_{C'}) = (1, 5)$. However, C' is not valid because it is not part of a cycle in H although $\{1, 5\} \in E(H)$. C' is shown in Figure 4.6.

- For C'' formed by $1, 6$ and 7, $v_{C''} = 7$, $u_{C''} = u_{G,7,\{1,6,7\}} = 1$, and $(v_{C''}, u_{C''}) = (7, 1)$. This is a valid pair since $(1, 7) \notin E(H)$, but C'' is a cycle in the graph resulting from H by adding the edge $\{1, 7\}$.

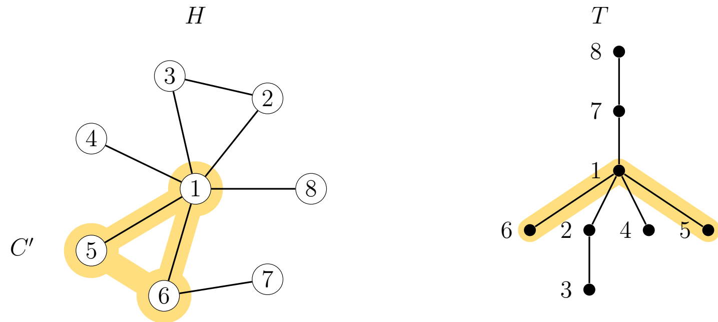

Figure 4.6: The cycle C' in H and T.

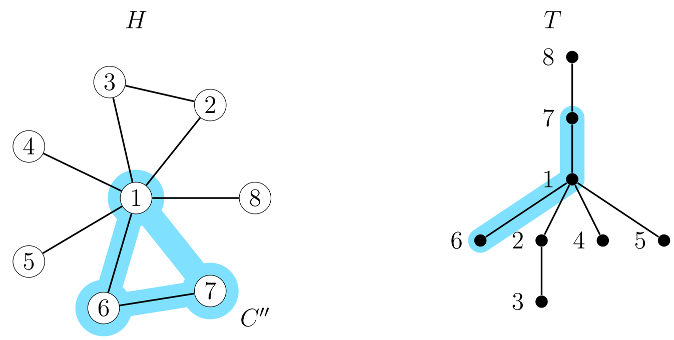

Figure 4.7: The cycle C'' in H and T.

Finally, since $(1, 2) <_{\text{lex}} (7, 1)$, then $(v, u) = (1, 2)$ and $\lambda(H, T) = (H^{-}, T)$, where $E(H^{-}) = E(H) \setminus \{\{1, 2\}\}$.

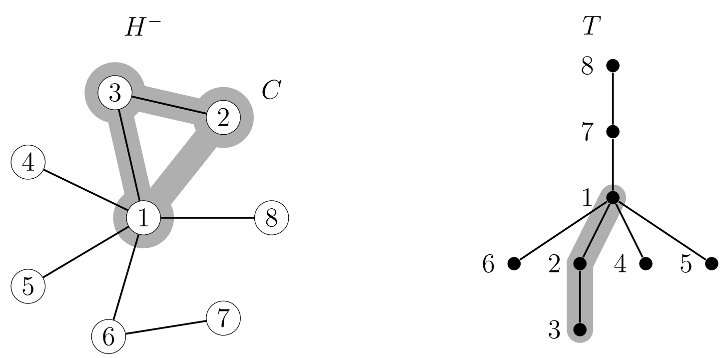

Figure 4.8: The cycle C gives $\lambda(H, T) = (H^{-}, T)$.

Note that by definition, the fixed points of λ are precisely those (H, T) for which it is not possible to find the pair $(v, u)_{H,T}$, i.e., $W_{H,T} = \emptyset$. We can characterize them using the full spanning trees of Proposition 4.4 associated with μ-trees.

Lemma 4.7. Let $(H, T) \in \Lambda_G$ be a fixed point of λ. Then

1. H is a full spanning tree of G.

2. The full spanning tree $\Psi = \Psi(G, T)$ given in Proposition 4.4 satisfies $\lambda(\Psi, T) = (\Psi, T)$.

3. $H = \Psi$.

Thus, the fixed points of λ are of the form $(\Psi(G, T), T)$, where T is a $\mu(G)$-tree.

Proof. 1. If H is not a spanning tree of G, there is a cycle C in H. Being T a $\mathcal{B}(H)$-tree, there must be a vertex $v \in [n]$ such that $V(C) \subseteq T_{\leq v}$. Otherwise, there would be $i_1, \ldots, i_k \in [n]$ such that $V(C) \subseteq T_{\leq i_1} \cup \cdots \cup T_{\leq i_k}$, but $V(C) \nsubseteq T_{\leq i_j}$, for any j. This would imply, by (B1), that $T_{\leq i_1} \cup \cdots \cup T_{\leq i_k} \in \mathcal{B}(H)$, contradicting (BF2). In conclusion, $V(C) \subseteq T_{\leq v}$ for some v, and thus, $W_{H,T} \neq \emptyset$. By construction, $\lambda(H, T) \neq (H, T)$ as required.

2. Proposition 4.4 shows that $\Psi = \Psi(G, T)$ is a full spanning tree for G and T is a $\mathcal{B}(\Psi)$-tree, i.e, $(\Psi, T) \in \Lambda_G$. If $\lambda(\Psi, T) \neq (\Psi, T)$, since Ψ has no cycles, then the only possibility is that $\lambda(\Psi, T) = (\Psi^+, T)$, where $E(\Psi^+) = E(\Psi) \cup \{v, u\}$, and $(v, u) = (v_C, u_C)$ for a cycle C in Ψ^+ formed when $\{v, u\}$ is added to Ψ. Moreover, $V(C) \subseteq T_{\leq v}$, and $u = u_{G,v,V(C)}$.

   Let $u_0 \in V(C)$ be such that $\{v, u_0\} \in E(\Psi)$. By the choice of u, then $u \leq u_0$ in $\mathbb{N}$. We claim that if $a \lessdot v$ is such that $u_0 \in T_{\leq a}$, then $u \in T_{\leq a}$. Assuming this for a moment, we get by the construction of Ψ that $u_0 = u_{G,v,T_{\leq a}}$, so $u_0 \leq u$ in $\mathbb{N}$. Therefore, $u_0 = u$, implying that $\{v, u\} \in E(\Psi)$ which is a contradiction. In concluison, $\lambda(\Psi, T) = (\Psi, T)$.

   To prove the claim, suppose by contradiction that there is another $b \lessdot v$ such that $u \in T_{\leq b}$. Then $V(C) \subseteq T_{\leq a} \cup T_{\leq b} \cup \{v\}$. Note that $V(C) \setminus \{v\} \in \mathcal{B}(\Psi^+)$ because $E(C) \setminus \{\{v, u\}, \{v, u_0\}\}$ is still a connected path from u to u_0. Thus, $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(\Psi^+)$ contradicting (BF2) since T is $\mathcal{B}(\Psi^+)$-tree, thanks to Lemma 4.6(2).

3. Assume by contradiction that $H \neq \Psi$. By interchanging H and Ψ we can assume that there is $\{v, u\} \in E(\Psi) \setminus E(H)$. Let $H_0 \preceq G$ such that $E(H_0) = E(H) \cup \{\{v, u\}\}$. Using H_0 we will prove that $\lambda(H, T) \neq (H, T)$, contrary to the assumption.

   Let us see that T is a $\mathcal{B}(H_0)$-tree. First, (BF1) holds since $\mathcal{B}(H) \subseteq \mathcal{B}(H_0)$. Second, if (BF2) is not valid, as in the proof of Lemma 4.6(2), there are $a, b \in [n]$ non-comparable in T such that $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(H_0) \setminus \mathcal{B}(H)$, $v \in T_{\leq a}$ and $u \in T_{\leq b}$. But T is a $\mathcal{B}(\Psi)$-tree, so $T_{\leq a}, T_{\leq b} \in \mathcal{B}(\Psi)$ and thus $T_{\leq a} \cup T_{\leq b} \in \mathcal{B}(\Psi)$ because $\{v, u\} \in E(\Psi)$. This contradicts (BF2) for T as a $\mathcal{B}(\Psi)$-tree. In conclusion, T is a $\mathcal{B}(H_0)$-tree as required.

   Now, since H is a full spanning tree of G, the edge $\{v, u\}$ is part of a cycle C of H_0. By the construction on Ψ we can assume that $u \in T_{\leq v}$, say $u = u_{G,v,T_{\leq a}}$, for some child a of v in T. Since T is a $\mathcal{B}(H_0)$-tree and $T_{\leq v} \in \mathcal{B}(H)$, then $V(C) \subseteq T_{\leq a} \cup \{v\} \subseteq$

$T_{\leq v}$. From here it follows that $u = u_C = u_{G,v,V(C)}$. Indeed, since $u = u_{G,v,T_{\leq a}}$ and $u_C \in V(C) \setminus \{v\} \subseteq T_{\leq a}$, then $u \leq u_C$ in $\mathbb{N}$. Conversely, by definition $u_C = u_{G,v,V(C)}$ and $\{v, u\} \in E(C)$, so $u_C \leq u$ in $\mathbb{N}$. Therefore, $(v, u) = (v_C, u_C) \in W_{H,T}$ proving that $\lambda(H, T) \neq (H, T)$ as we wanted.

□

Lemma 4.8. Let G be a connected graph on $[n]$. Then $\lambda : \Lambda_G \to \Lambda_G$ is an involution that reverses the sign of $(-1)^{|E(H)|} t^{\operatorname{des} T}$, for non-fixed points $(H, T) \in \Lambda_G$ of λ.

Proof. To show that λ is an involution, we consider the three possible cases according to its definition:

(I) If $\{v, u\} \in E(H)$, then $\lambda(H, T) = (H^-, T)$, where $E(H^-) = E(H)\setminus\{\{v, u\}\}$. By construction, $(v, u) = (v_C, u_C)$ for a cycle C in H, where $u \in V(C) \subseteq T_{\leq v}$. Then $(v, u) \in W_{H^-,T}$ since (ii) holds for H^-, i.e., $\{v, u\} \notin E(H^-)$ but C is a cycle in H. Noticing that $W_{H^-,T} \subseteq W_{H,T}$, it follows that $(v, u) = \min_{\text{lex}} W_{H,T} \leq_{\text{lex}} \min_{\text{lex}} W_{H^-,T} = (v, u)_{H^-,T}$. Therefore, $(v, u) = (v, u)_{H^-,T}$, and by definition of λ we obtain $\lambda(H^-, T) = (H, T)$ as required.

(II) If $\{v, u\} \notin E(H)$, $\lambda(H, T) = (H^+, T)$ where $E(H^+) = E(H) \cup \{v, u\}$. In this case, $(v, u) = (v_C, u_C)$ for a cycle C in H^+ and $u \in V(C) \subseteq T_{\leq v}$.

Note that $(v, u) \in W_{H^+,T}$, thus it is nonempty. Then $\min_{\text{lex}} W_{H^+,T} = (v_D, u_D)$ exists, where D is some cycle in G such that $u_D \in V(D) \subseteq T_{\leq v_D}$. In particular,

$$(v_D, u_D) \leq_{\text{lex}} (v, u).$$

We will prove that either $\{v, u\} \notin E(D)$ nor $\{v, u\} \in E(D)$ imply that $(v_D, u_D) \in W_{H,T}$. This amounts to saying that $(v, u) \leq_{\text{lex}} (v_D, u_D)$ and thus $(v_D, u_D) = (v, u)$. Consequently, $\lambda(H^+, T) = (H, T)$ as needed.

To conclude this case, we consider the above possibilities:

a) $\{v, u\} \notin E(D)$. In the case (i) when D is a cycle in H^+, then the same is valid for H as $E(H) = E(H^+) \setminus \{\{v, u\}\}$. On the other hand, if (ii) holds, $\{v_D, u_D\} \notin E(H^+)$, but D is a cycle in the graph with edges $E(H^+) \cup \{\{v_D, u_D\}\}$. By the same reason as before, this is also true for H. In both cases (i) and (ii) we see that $(v_D, u_D) \in W_{H,T}$.

b) $\{v, u\} \in E(D)$. Let Q be the cycle in G given by $E(Q) = E(C) \cup E(D) \setminus \{\{v, u\}\}$. Since $V(Q) \subseteq T_{\leq v} \cup T_{\leq v_D}$ and $v \in V(D) \subseteq T_{\leq v_D}$, then $V(C) \subseteq T_{\leq v} \subseteq T_{\leq v_D}$ and so $V(Q) \subseteq T_{\leq v_D}$. Moreover, Q contains the edge $\{v_D, u_D\}$. Therefore, $u_D \in V(Q) \subseteq T_{\leq v_D}$ and thus $v_Q = v_D$. We will see that $u_D = u_Q$. First, by definition of u_Q we have also $u_Q \leq u_D$ in $\mathbb{N}$, so we are left to prove the reverse inequality. We have two cases: if $u_Q \in V(D)$, by definition of u_D we have that $u_D \leq u_Q$ in $\mathbb{N}$ and we are done. In the second case, $u_Q \in V(C)$. Here we have also two cases: (a) If $\{v_D, u_Q\} = \{v_Q, u_Q\} \in E(Q) \setminus E(C) = E(D) \setminus \{\{v, u\}\}$, again by the definition of u_D we conclude $u_D \leq u_Q$ in $\mathbb{N}$ as needed. (b) if $\{v_D, u_Q\} \in E(C)$, since C is a cycle in H^+, $V(C) \subseteq T_{\leq v_D}$ and we know from the previous paragraph that $v_D \leq v$ (and also $u_D \leq u$) in $\mathbb{N}$, it must be that $v_D = v$. By definition of $u = u_C$ we have that $u \leq u_Q$ in $\mathbb{N}$, so by transitivity, $u_D \leq u_Q$ in $\mathbb{N}$ as needed. In conclusion,

$$(v_D, u_D) = (v_Q, u_Q).$$

In the case (i) when D is a cycle in H^+, we see that Q is a cycle in H. When (ii) holds and D is a cycle in the graph with edges $E(H^+) \cup \{\{v_D, u_D\}\}$, then Q is a cycle in the graph with edges $E(H) \cup \{\{v_D, u_D\}\}$. In both cases (i) and (ii) we see that Q is a cycle giving $(v_D, u_D) = (v_Q, u_Q) \in W_{H,T}$, as it was required.

(III) If (H,T) is fixed by λ, then $\lambda \circ \lambda(H,T) = (H,T)$.

In any case, $\lambda \circ \lambda = \mathrm{id}_{\Lambda_G}$, thus the map is an involution. Finally, if (H,T) is not fixed by λ, it is clear that λ reverses the sign of $(-1)^{|E(H)|} t^{\operatorname{des} T}$. Indeed, if $\lambda(H,T) = (H^{\pm}, T)$, then $|E(H^{\pm})| = |E(H)| \pm 1$. □

THEOREM 4.9. If G is a finite graph on $[n]$, the $\mu(G)$-polynomial is given by the formula

$$\mu_G(t) = (-1)^{V(G)-k(G)} \cdot \sum_{T \ \mu(G)\text{-forest}} t^{\ \operatorname{des} T}, \tag{4.1}$$

where the sum is taken over all μ-forests of G.

Proof. As usual, it is sufficient to prove the result for G connected. Assuming this and applying the formulas in Theorem 3.3 and Proposition 3.2, we can write

$$\mu_G(t) = \sum_{H \preceq G} (-1)^{|E(H)|} h_{\mathcal{B}(H)}(t) = \sum_{(H,T) \in \Lambda_G} (-1)^{|E(H)|} t^{\ \operatorname{des} T}.$$

By Lemma 4.8, $\lambda : \Lambda_G \setminus \mathrm{Fix}(\lambda) \to \Lambda_G \setminus \mathrm{Fix}(\lambda)$ is a sign-reversing involution, where $\mathrm{Fix}(\lambda)$ is the set of fixed points of this map. Moreover, Lemma 4.7 proves that $\mathrm{Fix}(\lambda) = \{(\Psi(G,T), T) \in \Lambda_G : T \text{ is a } \mu(G)\text{-tree}\}$. Therefore,

$$\mu_G(t) = \sum_{(H,T) \in \mathrm{Fix}(\lambda)} (-1)^{n-1} t^{\operatorname{des} T} = (-1)^{n-1} \cdot \sum_{T \ \mu(G)\text{-tree}} t^{\operatorname{des} T},$$

as we wanted to prove. □

EXAMPLE 4.3.2. Recalling Example 4.1.1, since $\operatorname{des} Y_1 = 0$ and $\operatorname{des} Y_2 = 1$, then $\mu_{P_2}(t) = (-1)^{2-1}(t^{\operatorname{des} Y_2} + t^{\operatorname{des} Y_1}) = -(t+1)$ as expected.

For the case of C_3, note that $\operatorname{des} T_1 = 0$, $\operatorname{des} T_2 = 1$, $\operatorname{des} T_3 = 0$, $\operatorname{des} T_4 = 1$, $\operatorname{des} T_5 = 1$, $\operatorname{des} T_6 = 1$, $\operatorname{des} T_7 = 1$, $\operatorname{des} T_8 = 2$, and $\operatorname{des} T_9 = 2$. Thus, we recover the polynomial

$$\mu_{C_3}(t) = (-1)^{3-1}\left(t^0 + t^1 + t^0 + t^1 + t^1 + t^1 + t^1 + t^2 + t^2\right) = 2 + 5t + 2t^2.$$

Thank to formula (4.1) we are able to establish the degree of the μ-polynomials and the sign of its coefficients, as it was announced in Remark 2.1.1.

Corollary 4.10. Let G be a finite graph. Then, the polynomial $(-1)^{V(G)-k(G)}\mu_G(t)$ has degree $V(G) - k(G)$ and has non-negative coefficients. In particular, if G is a connected graph on $[n]$, $|\mu|_G(t) = (-1)^{n-1}\mu_G(t)$ has degree $n-1$ and non-negative coefficients.

Proof. It is sufficient to prove the result when G is a connected graph on $[n]$. Since $n-1$ is the maximal number of possible descents of a tree on $[n]$, it is enough to exhibit a $\mu(G)$-tree T on $[n]$ with $\mathrm{des}(T) = n-1$. To do so, choose n (the maximal element of $[n]$) as root of T and consider the connected components $G'_1, \ldots, G'_r$ of $G|_{[n-1]}$. By construction, $\pi = \{V(G'_1), \ldots, V(G'_r)\}$ is a n-admissible partition. Then, choose in each $V(G'_j)$ the maximal element v_j (w.r.t. the order of $\mathbb{N}$) and connect n with $v_1, \ldots, v_r$. Repeating the procedure, we obtain the desired $\mu(G)$-tree T, due to Proposition 4.1. We conclude that the leading coefficient of $(-1)^{n-1}\mu_G(t)$ is at least 1, thus the conclusion. Finally, formula (4.1) shows that all the coefficients of $\mu_G(t)$ have the same sign as required. □

4.4 An application to kite-like graphs

We conclude this chapter with an application of μ-trees to compute explicitly a new example of the μ-polynomials of a family of graphs which we call kite-like due to their shape.

Example 4.4.1 (Kite-like graph). Let U_n be the graph on $[n]$ displayed in Figure 4.9 for $n \geq 4$.

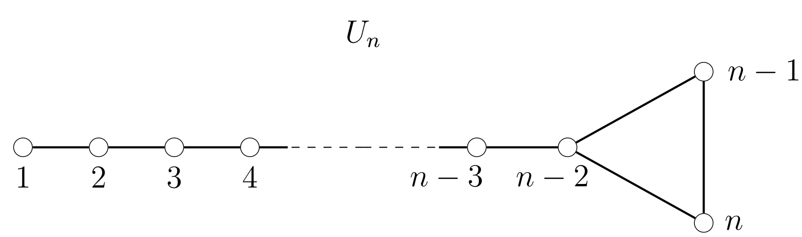

Figure 4.9: The graph U_n.

We can determine an explicit formula for $\mu_{U_n}(t)$ analyzing the structure of the $\mu(U_n)$-trees T. First, Theorem 4.9 shows that $|\mu|_{U_n}(t) = \sum_{T\ \mu(U_n)\text{-tree}} t^{\mathrm{des}\, T}$ is the (signless) μ-polynomial associated to U_n. Second, following the algorithm described in Proposition 4.2, after selecting the root $v \in [n]$ of T, we have the following five cases:

1. If $v = 1$, $U_n|_{[n]\setminus\{1\}} = U_{n-1}$ and the only 1-admissible partition is $\pi = \{\{2, \ldots, n\}\}$. These trees are then in correspondence with $\mu(U_{n-1})$-trees with the same number of descents, since $1 < u$, for all $u = 2, \ldots, n$. Thus, $\nu = 1$ produces the term $|\mu_{U_{n-1}}|(t)$.

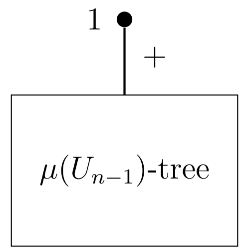

Figure 4.10: $\mu(U_n)$-trees with root $v = 1$.

2. If $v = 2, \ldots, n-3$, $U_n|_{[n]\setminus\{v\}}$ is formed by P_{v-1} on $\{1, 2, \ldots, v-1\}$ and U_{n-v} on $\{v+1, v+2, \ldots, n\}$ as connected components. The only v-admissible partition is $\pi = \{\{1, 2, \ldots, v-1\}, \{v+1, v+2, \ldots, n\}\}$ since v is only connected by edges in U_n with $v-1$ and $v+1$.

   If $u = 1, 2, \ldots, v-1$, then $v > u$, and v gives an extra descent for the corresponding $\mu(P_{v-1})$-trees, thus inducing the term $t\mathcal{N}_{v-1}(t)$. If $u = v+1, \ldots, n$, $v < u$, v gives no additional descent, and we get $|\mu|_{U_{n-v}}(t)$. Therefore, the $\mu(U_n)$-trees rooted at v produce the term $t\mathcal{N}_{v-1}(t)|\mu|_{U_{n-v}}(t)$ in $|\mu|_{U_n}(t)$.

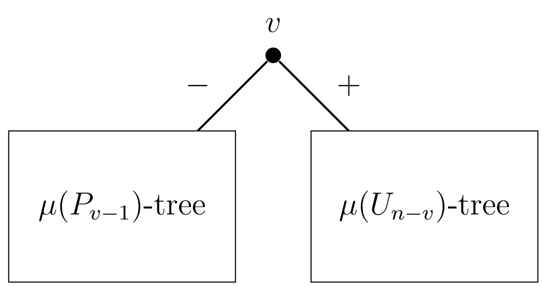

Figure 4.11: $\mu(U_n)$-trees with root $v = 2, \ldots, n-3$.

3. If $v = n-2$, $U_n|_{[n]\setminus\{n-2\}}$ has P_{n-3} and P_2 as connected components. The $(n-2)$-admissible partitions are $\{\{1, 2, \ldots, n-3\}, \{n-1, n\}\}$ and $\{\{1, 2, \ldots, n-3\}, \{n-1\}, \{n\}\}$. The first partition gives $t\mathcal{N}_2(t)\mathcal{N}_{n-3}(t) = t(t+1)\mathcal{N}_{n-3}(t)$, since the factor t comes from the extra descent induced by $v = n-2$. In the same way, the second partition gives $t\mathcal{N}_{n-3}(t)$. In total, the $\mu(U_n)$-trees rooted at $n-2$ give the term $t(t+2)\mathcal{N}_{n-3}(t)$.

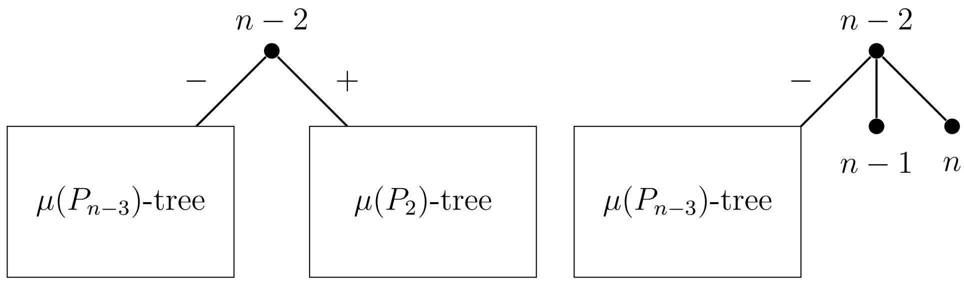

Figure 4.12: $\mu(U_n)$-trees with root $v = n-2$.

4. If $v = n-1$, $U_n|_{[n]\setminus\{n-1\}} = P_{n-1}$ and the $(n-1)$-admissible partitions are $\{\{1, 2, \ldots, n-2\}, \{n\}\}$ and $\{\{1, 2, \ldots, n-2, n\}\}$. The firs partition gives the term $t\mathcal{N}_{n-2}(t)$. For the second one, we would get the term $t\mathcal{N}_{n-1}(t)$ if all elements of $\{1, 2, \ldots, n-2, n\}$ were less than $n-1$. However, this fails for $u = n$. Therefore, the correct term is

$$t\left(\mathcal{N}_{n-1}(t) - t\mathcal{N}_{n-2}(t)\right) + t\mathcal{N}_{n-2}(t),$$

where $\mathcal{N}_{n-1}(t) - t\mathcal{N}_{n-2}(t)$ stands for the $\mu(P_{n-1})$-trees on $\{1, 2, \ldots, n-2, n\}$ except the ones with root at $u = n$, and the term $t\mathcal{N}_{n-2}(t)$ compensates the $\mu(U_{n-1})$-trees rooted at $u = n$ with the descent counted correctly. In total, the term obtained in this case is $t\mathcal{N}_{n-1}(t) + (2t - t^2)\mathcal{N}_{n-2}(t)$.

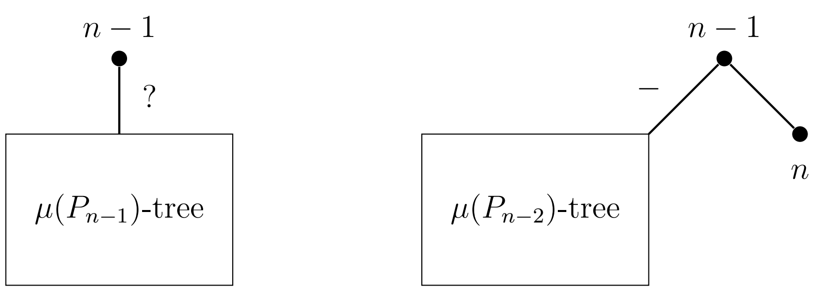

Figure 4.13: $\mu(U_n)$-trees with root $v = n-1$.

5. If $v = n$, the n-admissible partitions for the restricted graph $U_n|_{[n]\setminus\{n\}} = P_{n-1}$ are $\{\{1, 2, \ldots, n-2\}, \{n-1\}\}$ and $\{\{1, 2, \ldots, n-2, n-1\}\}$. The first partition gives the term $t^2\mathcal{N}_{n-2}(t)$ due to the two descents produced by $v = n$. Finally, for the second one, we get $t\mathcal{N}_{n-1}(t)$. In total, this case gives $t\mathcal{N}_{n-1}(t) + t^2\mathcal{N}_{n-2}(t)$.

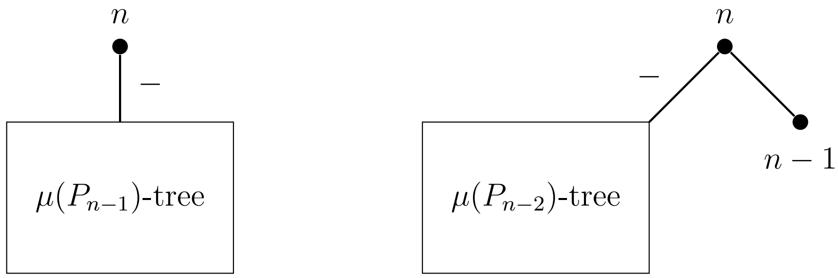

Figure 4.14: $\mu(U_n)$-trees with root $v = n$.

The previous analysis allows to conclude that

$$\begin{aligned}|\mu|_{U_n} =&|\mu|_{U_{n-1}} + t\sum_{j=2}^{n-3}\mathcal{N}_j|\mu|_{U_{n-1-j}} + t(t+2)\mathcal{N}_{n-3}\\ &+ t\mathcal{N}_{n-1} + (2t-t^2)\mathcal{N}_{n-2} + t\mathcal{N}_{n-1} + t^2\mathcal{N}_{n-2}.\end{aligned}$$

To unify notation set $\mu_{U_1}(t) = \mu_{P_1}(t) = 1$, $\mu_{U_2}(t) = \mu_{P_2}(t) = -(t+1)$ and $\mu_{U_3}(t) = \mu_{C_3}(t) = 2t^2 + 5t + 2$. Also let $\mathcal{N}_0(t) = 1/t$ and $|\mu|_{U_0}(t) = 0$. Then, the previous recurrence takes the form

$$|\mu|_{U_n} = t\sum_{j=0}^{n-1}\mathcal{N}_j|\mu|_{U_{n-1-j}} + 2t\mathcal{N}_{n-1} + t\mathcal{N}_{n-2} + t\mathcal{N}_{n-3}, \qquad n \geq 3. \tag{4.2}$$

We can solve this by using generating series. First, if we let $\mu(z,t) = \sum_{n=0}^{\infty}\mathcal{N}_n(t)z^n$, we have from the series (2.8) that $t\mu - 1 = \frac{1-z(t+1)-\sqrt{R}}{2z} = \frac{1-z-\sqrt{R'}}{2z} - \frac{t}{2}$ and $1 - tz\mu = \frac{1+z(t-1)+\sqrt{R'}}{2}$, where $R' = 1 - 2z(t+1) + z^2(1-t)^2$. In particular,

$$\begin{aligned}\frac{1}{1-tz\mu} &= \frac{2}{1+z(t-1)+\sqrt{R'}} \\ &= \frac{1-z-\sqrt{R'}}{2tz} + \frac{1}{2} = 1 + \mu - \frac{1}{t} = 1 + \sum_{n=1}^{\infty}\mathcal{N}_n z^n.\end{aligned} \tag{4.3}$$

Now, (4.2) means that the series $\mathcal{U}(z,t) = \sum_{n=1}^{\infty}|\mu|_{U_n}(t)z^n$ satisfies

$$\begin{aligned}\mathcal{U} =&z + (t+1)z^2 + \sum_{n=3}^{\infty}(2t\mathcal{N}_{n-1} + t\mathcal{N}_{n-2} + t\mathcal{N}_{n-3})z^n + t\sum_{n=3}^{\infty}\sum_{j=0}^{n-1}\mathcal{N}_j|\mu_{U_{n-j-1}}|z^n\\ =&z + (t+1)z^2 + 2tz(\mu - \mathcal{N}_0 - z) + tz^2(\mu - \mathcal{N}_0) + tz^3\mu\\ &+ \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\mathcal{N}_j|\mu|_{U_{n-j-1}}z^n - t(\mathcal{N}_0|\mu_{U_1}|)z^2\\ =&z + (t+1)z^2 + 2z(t\mu - 1 - zt) + z^2(t\mu - 1) + tz^3\mu + tz\mu\mathcal{U} - z^2\\ =&2 - tz^2 + (2 + z + z^2)(tz\mu - 1) + tz\mu\mathcal{U}.\end{aligned}$$

Thanks to (4.3), this shows that $\mathcal{U}(z,t)$ is given by

$$\begin{aligned}\mathcal{U} &= -(2+z+z^2) + (2-tz^2)\left(1+\sum_{n=1}^{\infty}\mathcal{N}_n z^n\right)\\ &= -z-(1+t)z^2 + 2\sum_{n=1}^{\infty}\mathcal{N}_n z^n - t\sum_{n=3}^{\infty}\mathcal{N}_{n-2}z^n\\ &= z+(1+t)z^2+\sum_{n=3}^{\infty}(2\mathcal{N}_n - t\mathcal{N}_{n-2})z^n.\end{aligned}$$

Finally, recalling the relation (2.7), we have proved the following result.

THEOREM 4.11. We have that

$$(-1)^{n-1}\mu_{U_n}(t) = 2\mathcal{N}_n(t) - t\mathcal{N}_{n-2}(t), \qquad n \geq 3, \tag{4.4}$$

where $\mathcal{N}_n(t)$ are the Narayana polynomials.

The first values of these polynomials are

$$\begin{aligned}|\mu|_{U_4}(t) &= 2t^3+11t^2+11t+2,\\ |\mu|_{U_5}(t) &= 2t^4+19t^3+37t^2+19t+2,\\ |\mu|_{U_6}(t) &= 2t^5+29t^4+94t^3+94t^2+29t+2,\\ |\mu|_{U_7}(t) &= 2t^6+41t^5+200t^4+330t^3+200t^2+41t+2.\end{aligned}$$

Note that in general $|\mu|_{U_n}(0) = 2\mathcal{N}_n(0) = 2$.

5

Remarks on real-rootedness

Many families of univariate polynomials encountered in various areas of mathematics, statistics and computer science are known or conjectured to have real roots. One of their features is that they determine if a finite sequence is unimodal or log-concave, properties of utmost importance in Combinatorics, see, e.g., [22, 5]. On the other hand, the notion of interlacing polynomials is a stronger property that also appears in other contexts such as the study of orthogonal polynomials. In this final chapter we study these properties for the μ-polynomials of graphs and we establish them for the families of cyclic and kite-like graphs. In the first section we recall some tools on this theory and in the second one we establish the new results. For the case of the cyclic graphs this will done with the aid of classical Legendre polynomials and a new four-term recursion for the μ_{C_n}-polynomials. Finally, we include a section discussing the interlacing property relating graphs and its contractions by an edge.

5.1 Real-rooted polynomials and interlacing sequences

In this section we recall some definitions and results scattered in the literature on real-rooted and interlacing polynomials that will be useful for our purposes.

We recall that a polynomial $P(t) \in \mathbb{R}[t]$ is *real-rooted* if for all $r \in \mathbb{C}$ such that $P(r) = 0$, we have that $r \in \mathbb{R}$. In this case, if $\deg(P) = n$ and $r_1, \ldots, r_n$ denote its roots, we can sort them in increasing way, say, $r_1 \leq r_2 \leq \cdots \leq r_n$. Note that is P as non-negative coefficients (as it is the case of the polynomials $|\mu|_G$) and it is real-rooted, all its roots must be negative or zero, i.e., $r_n \leq 0$.

We also recall that if $Q \in \mathbb{R}[t]$ with $\deg(Q) = n-1$ is real-rooted with roots $s_1, \ldots, s_{n-1}$, it is said that Q *interlaces* P if

$$r_1 \le s_1 \le r_2 \le \cdots \le r_{n-1} \le s_{n-1} \le r_n. \tag{5.1}$$

If $\deg(Q) = n$, and

$$s_1 \le r_1 \le s_2 \le r_2 \le \cdots \le r_{n-1} \le s_n \le r_n,$$

it is said that Q *alternates* P (to the left). In both cases, we will denote these situations by $Q \preccurlyeq P$. Moreover, if the previous inequalities are strict, we say that Q strictly interlaces P or that Q strictly alternates f, respectively. This will be denoted by $Q \prec P$.

Example 5.1.1. The simplest example of interlacing polynomials is given by a real-rooted polynomial P and its derivative P'. In symbols, $P' \preceq P$. This follows at once from Rolle's theorem and Taylor expansions. Moreover, if P has degree n and only simple roots, say, $r_1 < r_2 < \cdots < r_n$, Rolle's theorem says precisely that there is $s_j \in (r_j, r_{j+1})$ such that $P'(s_j) = 0$, for $j = 1, \ldots, n-1$. Thus, P' is also real-rooted and (5.1) holds.

Example 5.1.2 (The $\mu(K_n)$-polynomials). The family of polynomials $\mu_{K_n}(t)$ is real-rooted thanks to Proposition 2.3. Moreover, $\mu_{K_n}(t) \prec \mu_{K_{n+1}}(t)$ for all $n \in \mathbb{N}^+$. Indeed, writing

$$s_j = \frac{j-n}{j}, \quad r_j = \frac{j-(n+1)}{j},$$

for the roots of $\mu_{K_n}(t)$ and $\mu_{K_{n+1}}(t)$, respectively, and in ascending order, it is clear that (5.1) holds with strict inequalities.

Example 5.1.3 (Orthogonal polynomials). A family of polynomials $\{Q_n(t)\}_{n\in\mathbb{N}}$, where $\deg(Q_n) = n$, is called *orthogonal* with respect to an non-negative integrable weight function $w : [a,b] \to \mathbb{R}$ if

$$\int_a^b Q_m(t)Q_n(t)w(t)dt = h_n\delta_{m,n}, \tag{5.2}$$

where $\delta_{m,n}$ is the usual Kroneker delta. The hypothesis on the degrees of these polynomials show that $\{Q_n(t)\}_{n\in\mathbb{N}}$ is a basis of $\mathbb{R}[t]$. This and the previous orthogonal relation prove that

$$\int_a^b x^m Q_n(t)w(t)dt = 0, \qquad \text{for all } m = 0, 1, \cdots, n-1, \tag{5.3}$$

since x^m can be written as a linear combination of $Q_0, Q_1, \ldots, Q_{n-1}$.

One important feature of orthogonal polynomials is that they satisfy a three-term relation of the form

$$Q_{n+1}(t) = (A_n t + B_n)Q_n(t) - C_n Q_{n-1}(t), \tag{5.4}$$

where $Q_{-1} \equiv 0$. Here A_n, B_n and C_n are real constants such that $A_n > 0$ and $C_n > 0$, see, e.g., [23, Theorem 3.2.1]. In turn, this recurrence allows to prove that

$$Q'_{n+1}(t)Q_n(t) - Q_{n+1}(t)Q'_n(t) > 0 \text{ for all } t \in \mathbb{R}, n \in \mathbb{N}, \tag{5.5}$$

see e.g. [2, Corollary 5.2.6], [23, p. 43].

Proposition 5.1. A sequence $\{Q_n(t)\}_{n\in\mathbb{N}}$ of orthogonal polynomials w.r.t. to a weight $w : [a, b] \to \mathbb{R}$ is real-rooted with simple zeros in $[a, b]$. Moreover, $Q_n \prec Q_{n+1}$, for all $n \in \mathbb{N}$.

Proof. Let t_0 be a zero of Q_n. Since $Q_n \in \mathbb{R}[t]$, then $\overline{t_0}$ is also a zero of Q_n, so $Q_n(t)/(t-\overline{t_0})$ is a polynomials of degree $n-1$. Then (5.3) shows that

$$0 = \int_a^b \frac{Q_n(t)}{t-\overline{t_0}} Q_n(t)w(t)dt = \int_a^b (t-t_0)\frac{|Q_n(t)|^2}{|t-t_0|^2} w(t)dt.$$

Therefore, $t_0 = \int_a^b t\frac{|Q_n(t)|^2}{|t-t_0|^2}w(t)dt / \int_a^b \frac{|Q_n(t)|^2}{|t-t_0|^2}w(t)dt \in [a, b]$. Moreover, t_0 must be simple. Otherwise,

$$0 = \int_a^b \frac{Q_n(t)}{(t-t_0)^2} Q_n(t)w(t)dt = \int_a^b \frac{|Q_n(t)|^2}{|t-t_0|^2} w(t)dt,$$

which is impossible since $w(t) \geq 0$ on $[a, b]$.

Now, let $a \leq \lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,n} \leq b$ be the roots of $Q_n(t)$. Then (5.5) shows that

$$Q'_{n+1}(\lambda_{n+1,k})Q_n(\lambda_{n+1,k}) > 0.$$

Thus $Q'_{n+1}(\lambda_{n+1,k})$ and $Q_n(\lambda_{n+1,k})$ have the same sign. Since the roots are simple, the values $Q'_{n+1}(\lambda_{n+1,k})$ and $Q'_{n+1}(\lambda_{n+1,k+1})$ have different sign. Therefore, $Q_n(\lambda_{n+1,k})$ and $Q_n(\lambda_{n+1,k+1})$ have also different sign. Thus, by the intermediate value theorem, Q_n must have a zero in $(\lambda_{n+1,k}, \lambda_{n+1,k+1})$. This proves that $Q_n(t)$ interlaces $Q_{n+1}(t)$. □

Example 5.1.4 (Legendre polynomials). The Legendre polynomials are the orthogonal ones corresponding to the constant weight $w(t) \equiv 1$ on $[-1, 1]$. They are given explicitly by *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left[(1-t^2)^n\right].$$

In this case the three-term recurrence (5.4) takes the form

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t), \tag{5.6}$$

where $P_0(t) = 1$ and $P_1(t) = t$. The first few values are given by

$$\begin{aligned}
P_2(t) &= \tfrac{1}{2}\left(3x^2 - 1\right),\\
P_3(t) &= \tfrac{1}{2}\left(5x^3 - 3x\right),\\
P_4(t) &= \tfrac{1}{8}\left(35x^4 - 30x^2 + 3\right),\\
P_5(t) &= \tfrac{1}{8}\left(63x^5 - 70x^3 + 15x\right),\\
P_6(t) &= \tfrac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right).
\end{aligned}$$

Being orthogonal, by Proposition 5.1, they are real-rooted with simple zeros and $P_{n-1}(t) \prec P_n(t)$, as it is evidenced in Figure 5.1 for some values of n.

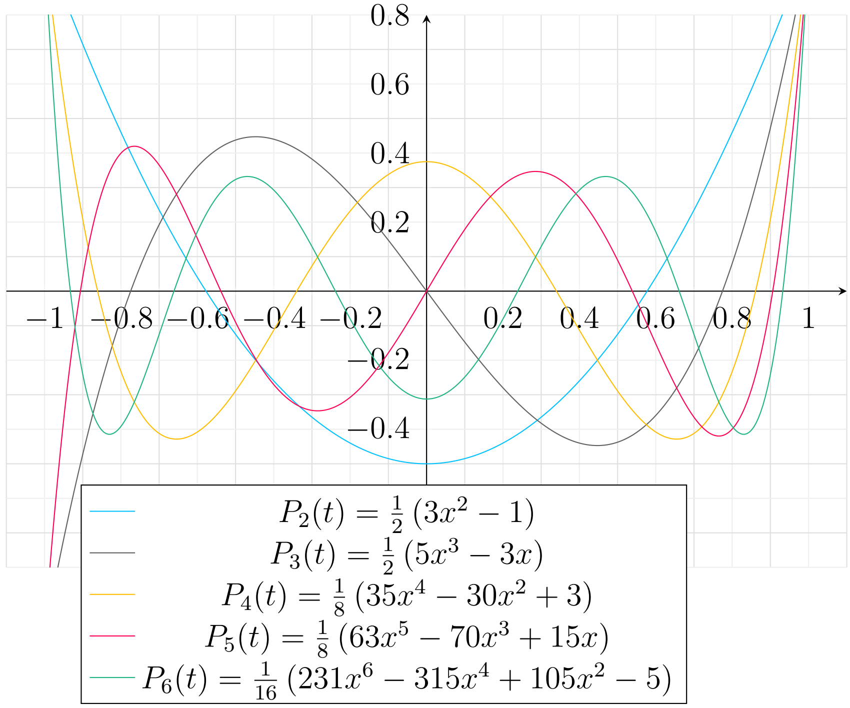

Figure 5.1: The first Legendre polynomials $P_n(t)$.

It is worth noticing that these polynomials are part of a larger family known as the *Jacobi polynomials of index* $(\alpha, \beta) \in (-1, +\infty)^2$. The former are the orthogonal polynomials associated to the weight function $w(t) = (1-t)^\alpha (1+t)^\beta$ on the interval $[-1, 1]$ and are given explicitly by

$$(1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[(1-t)^{n+\alpha} (1+t)^{n+\beta} \right],$$

see [2, Remark 2.5.1]. Therefore, the Legendre polynomials are the Jacobi polynomials of index $(0, 0)$. In this work we will only employ these and the polynomials $\{P_n^{(1,1)}(t)\}_{n \in \mathbb{N}}$ with weight $(1, 1)$.

We recall that a polynomial $P \in \mathbb{R}[t]$ is called *standard* if either it is identically zero or its leading coefficient is positive.

Let $\{Q_n(t)\}_{n \geq 0}$ be a sequence of standard polynomials. It is said that $\{Q_n(t)\}_{n \geq 0}$ is a *Sturm sequence* if:

1. $\deg Q_n = n$, for all n,
2. $Q_{n-1}(r) Q_{n+1}(r) < 0$ whenever $Q_n(r) = 0$ and $n \geq 1$.

It can be proved that $\{Q_n(t)\}_{n\geq 0}$ is a Sturm sequence if and only if, for $n \geq 1$, Q_n is real-rooted and $Q_n \prec Q_{n+1}$. The reader may consult [25, Section 4] for more information.

Among the results available to determine whether a sequence of polynomials is real-rooted or interlacing, we highlight the following. The proofs can be consulted in the references below.

Theorem 5.2 (c.f. Theorem 1 [26]). Let $P, Q \in \mathbb{R}[t]$ be polynomials whose leading coefficients have the same sign. Suppose that P and Q are real-rooted and that $Q \preccurlyeq P$. If $a, b, c, d \in \mathbb{R}$ satisfy $ad \geq bc$, then

$$(bt + a)P(t) + (dt + c)Q(t)$$

is real-rooted.

The following is a statement on interlacing sequences satisfying four-term recurrences which is proved in [13].

Theorem 5.3 (c.f. Theorem 2.3 [13]). Let $F, f, g_1, \ldots, g_k \in \mathbb{R}[t]$ be polynomials satisfying the following conditions:

1. $F(x) = a(x)f(x) + b_1(x)g_1(x) + \cdots + b_k(x)g_k(x)$, where $a, b_1, \ldots, b_k \in \mathbb{R}[t]$ and $\deg(F) = \deg(f)$ or $\deg(F) = \deg(f) + 1$.
2. $f, g_1, \ldots, g_k$ are real-rooted and $g_j \preceq f$, for all j.
3. F and $g_1, \ldots, g_k$ have leading coefficients of the same sign.

Suppose that $b_j(r) \leq 0$ for each j and each zero r of f. Then F is real-rooted and $f \preceq F$. In particular, if for each zero r of f, there is an index j such that $g_j \prec f$ and $b_j(r) < 0$, then $f \prec F$.

Theorem 5.4 (c.f. Corollary 2.4 [13]). Let $V_n(t)$ be a sequence of standard polynomials satisfying the recurrence relation

$$V_n(t) = a_n(t)V_{n-1}(t) + b_n(t)V'_{n-1}(t) + c_n(t)V_{n-2}(t), \tag{5.7}$$

where $a_n(t)$, $b_n(t)$, $c_n(t) \in \mathbb{R}[t]$. Suppose that for each n, the deg $V_n = n$ and coefficients of $V_n(t)$ are nonnegative. If either $b_n(t) < 0$ or $c_n(t) < 0$ whenever $t \leq 0$, then $\{V_n(t)\}_{n\in\mathbb{N}}$ is a Sturm sequence.

Example 5.1.5. The previous theorem allows to conclude again that a sequence of orthogonal polynomials $\{Q_n(t)\}_{n\in\mathbb{N}}$ conform a Sturm sequence thanks to the three-term recurrence (5.4). In fact, take $a_n(t) = A_n t + B_n$, $b_n(t) = 0$, and $c_n(t) = -C_n < 0$. Another interesting example is the sequence of Narayana polynomials $\{\mathcal{N}_n(t)\}_{n\in\mathbb{N}}$ which is known to satisfy the recurrence

$$(n + 1)\mathcal{N}_n(t) = (2n - 1)(1 + t)\mathcal{N}_{n-1} - (n - 2)(t - 1)^2\mathcal{N}_{n-2}(t). \tag{5.8}$$

By taking $a_n(t) = \frac{2n-1}{n+1}(1 + t)$, $b_n(t) = 0$, and $c_n(t) = -\frac{n-2}{n+1}(t - 1)^2 < 0$ (for all $t \neq 1$) in (5.7) we conclude this is also a Sturm sequence. There is yet another way to prove this assertion, as we shall see in the next section.

We conclude this section with some remarks on a change of variables that will be useful.

Remark 5.1.1. Consider the Möbius transformation

$$s(t) = \frac{1+t}{1-t}. \tag{5.9}$$

We can relate pairs of polynomials of the same degree m using the formulas

$$P(t) = (1-t)^m P^\dagger\left(\frac{1+t}{1-t}\right), \qquad P^\dagger(t) = \frac{(t+1)^m}{2^m} P\left(\frac{t-1}{t+1}\right).$$

In fact, given $P(t)$, the polynomial $P^\dagger(s)$ is uniquely determined by the previous expression, and it is obtained inverting $s(t)$. We have the following properties, which are immediate to check:

1. $P(t) = t^m P(1/t)$ if and only if $P^\dagger(-s) = (-1)^m P^\dagger(s)$.
2. $(P')^\dagger(s) = (s+1)(P^\dagger)' - mP^\dagger(s)$.
3. If $\deg(Q) = k \leq m$ and $\deg(P+Q) = m$, then $(P+Q)^\dagger(s) = P^\dagger(s) + \left(\frac{s+1}{2}\right)^{m-k} Q^\dagger(s)$. In particular, $(P+Q)^\dagger = P^\dagger + Q^\dagger$ if $k = m$ and $\deg(P+Q) = m$.

Moreover, s preserves real-rootedness and the interlacing property. In fact,

1. $P(t)$ is real-rooted if and only if $P^\dagger(s)$ is so, since $s(\mathbb{R}) = \mathbb{R}$.
2. Note that $s : (-\infty, 0) \to (-1, 1)$ is strictly increasing and bijective. Therefore, the roots of $P(t)$ lie on $(-\infty, 0)$ if and only if the roots of $P^\dagger(s)$ lie on $(-1, 1)$.
3. Assume that the roots of $P(t)$ and $Q(t)$ lie on $(-\infty, 0)$. Then, $P(s)$ (strictly) interlaces $Q(s)$ if and only if $P^\dagger(s)$ (strictly) interlaces $Q^\dagger(s)$.

5.2 Real-rootedness of cyclic and kite-like μ-polynomials

Numerical exploration for connected graphs G having up to 7 vertices has shown that μ_G is real-rooted having only simple roots, see Conjecture 2. The goal of this section is to verify this conjecture for the infinite families of cyclic graphs C_n and kite-like graphs U_n. As we saw in Example 5.1.2 the conjecture is true for the complete graphs K_n. It also holds for the path graphs P_n. In fact, P. Bränden showed in [4, p. 2] that

$$\mathcal{N}_n(t) = \frac{1}{n}(1-t)^{n-1} P_{n-1}^{(1,1)}\left(\frac{1+t}{1-t}\right), \tag{5.10}$$

where $P_n^{(1,1)}(t)$ is the nth Jacobi polynomial of index $(1,1)$, see Example 5.1.4 and Remark 5.2.1 below. Since $\{P_n^{(1,1)}(t)\}_{n\in\mathbb{N}}$ conforms a sequence of orthogonal polynomials w.r.t.

the weight $(1-t)(1+t)$ on $[-1,1]$, they are real-rooted and interlacing by Proposition 5.1. In terms of the transformation $s(t)$ given in Remark 5.1.1 equation (5.10) says that

$$\mathcal{N}_n(s) = \frac{1}{n}(P_{n-1}^{(1,1)})^{\dagger}.$$

Thus, using Proposition 2.5 we conclude that $\mu_{P_n}(t)$ is real-rooted and

$$\mu_{P_{n-1}} \prec \mu_{P_n}, \qquad \text{for all } n \geq 1.$$

We now turn to the cyclic polynomials. These also admit a similar representation in terms of Legendre polynomials. To proceed, we recall some other facts on these polynomials.

Remark 5.2.1. The Jacobi polynomials of index $(1,1)$ are determined by the generating series

$$\sum_{n=0}^{\infty} P_n^{(1,1)}(t) z^n = \frac{2}{R(1-zt+R)}, \qquad \text{where } R = \sqrt{1-2zt+z^2}, \tag{5.11}$$

see [2, Theorem 6.4.2]. Note that this expansion joint with (2.8) can be used to establish (5.10). On the other hand, the Legendre polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ have as generating series

$$\sum_{n=0}^{\infty} P_n(t) z^n = \frac{1}{\sqrt{1-2tz+z^2}}. \tag{5.12}$$

Among the numerous formulas they satisfy, we highlight the following:

$$P'_{n+1}(t) = (n+1)P_n(t) + tP'_n(t), \tag{5.13}$$

$$(t^2-1)P''_{n+1}(t) = (n+1)(n+2)P_{n+1}(t) - 2tP'_{n+1}(t), \tag{5.14}$$

$$(t^2-1)P'_n(t) = n(tP_n(t) - P_{n-1}(t)), \qquad n \geq 1, \tag{5.15}$$

see [23, Eq. 4.7.28]. Moreover, Legendre and Jacobi polynomials are related by

$$P'_n(t) = \frac{1}{2}(n+1)P_{n-1}^{(1,1)}(t), \tag{5.16}$$

see, e.g., [2, Eq. 6.3.8].

Proposition 5.5. The $\mu(C_n)$-polynomials of the cyclic graphs can be written in terms of the Legendre polynomials as

$$\mu_{C_n}(t) = (-1)^{n-1}(1-t)^{n-1}\left[\frac{2}{n+1}P'_n\left(\frac{1+t}{1-t}\right) - P_{n-1}\left(\frac{1+t}{1-t}\right)\right], \qquad n \geq 2.$$

Proof. To establish the result notice that $\mathcal{W}(z,t)$ in (2.12) can be written as

$$\mathcal{W}(z,t) = 1\Big/\sqrt{1-2z(1-t)\frac{t+1}{1-t} + [z(1-t)]^2}.$$

It follows from the expansion (5.12) that

$$W_n(t) = (1-t)^n P_n\left(\frac{1+t}{1-t}\right). \tag{5.17}$$

Equations (5.17), (5.10), and (5.16) establish the required formula. □

Note that formula (5.17) and the Proposition 5.1 also allow to conclude the following statement which is a known fact.

Corollary 5.6. The polynomials $W_n(t)$ (Narayana of type B) are real-rooted having simple zeros and $W_{n-1}(t) \prec W_n(t)$.

It turns out that the family of $\mu(C_n)$-polynomials for the cyclic graphs C_n satisfies a four-term recurrence. This was inspired in the closely related formulas for generalized Narayana polynomials given in [7]. The proof will be based on the Möbius transformation of Remark 5.1.1 and the properties of Legendre polynomials discused above.

Theorem 5.7. The cyclic polynomials satisfy the four-term recurrence

$$\begin{aligned}-n(n+3)\mu_{C_{n+2}}(t) = &2((n^2+3n+1)t+(n+1)^2)\mu_{C_{n+1}}(t)+n(n+1)(t-1)^2\mu_{C_n}(t)\\ &+2t(1-t)\mu'_{C_{n+1}}(t),\end{aligned}$$

valid for $n \geq 2$.

Proof. Letting $Q_n(t) = (-1)^{n-1}\mu_{C_n}(t)$, the recurrence asserts that

$$\begin{aligned}n(n+3)Q_{n+2}(t) = &2((n^2+3n+1)t+(n+1)^2)Q_{n+1}(t)-n(n+1)(t-1)^2Q_n(t)\\ &+2t(1-t)Q'_{n+1}(t).\end{aligned}$$

If $F_n = Q_n^\dagger$, replacing $Q_n(t) = (1-t)^{n-1}F_n(s)$, where $s = (1+t)/(1-t)$, and canceling $(1-t)^{n+1}$ we find that

$$\begin{aligned}n(n+3)F_{n+2}(s) = &2((n^2+3n+1)t+(n+1)^2)\frac{1}{1-t}F_{n+1}(s)-n(n+1)F_n(s)\\ &+\frac{2t}{1-t}(Q'_{n+1})^\dagger(s).\end{aligned}$$

Since $t = (s-1)/(s+1)$ and $(Q'_{n+1})^\dagger(s) = (s+1)F'_{n+1}(s) - nF_n(s)$, we see that the F_n must satisfy

$$n(n+3)F_{n+2}(s) = 2(n+1)^2sF_{n+1}(s) - n(n+1)F_n(s) + (s^2-1)F'_{n+1}(s). \tag{5.18}$$

Thus, proving the theorem is equivalent to prove (5.18). By Proposition 2.9 we have that

$$F_n = 2\frac{P'_n}{n+1} - P_{n-1}, \qquad n \geq 2.$$

Therefore, the left-hand side of (5.18) is equal to $n(n+3)F_{n+2} = n(n+1)P_{n+1} + 2nsP'_{n+1}$, thanks to (5.13). The right-hand side is equal to

$$\begin{aligned}&2(n+1)^2sF_{n+2}+(s^2-1)\left[2\frac{P''_{n+1}}{n+2}-P'_n\right]-n(n+1)F_{n+1}\\ &=2(n+1)^2s\left[2\frac{P'_{n+1}}{n+2}-P_n\right]+\frac{2}{n+2}\left[(n+1)(n+2)P_{n+1}-2sP'_{n+1})\right]-(s^2-1)P'_n\\ &\quad -n(2P'_n-(n+1)P_{n-1})\\ &=4nsP'_{n+1}-2(n+1)^2sP_n-(s^2-1+2n)P'_n+n(n+1)P_{n-1}+2(n+1)P_{n+1},\end{aligned}$$

where in the second equality we used (5.14). In this way, using (5.6) and (5.13), the difference between the right-hand side and the left-hand side of (5.18) is

$$
\begin{aligned}
&2nsP'_{n+1} - 2(n+1)^2sP_n - (s^2-1+2n)P'_n + n(n+1)P_{n-1} - (n+1)(n-2)P_{n+1}\\
=&2ns((n+1)P_n + sP'_n) - 2(n+1)^2sP_n - (s^2-1+2n)P'_n + n(n+1)P_{n-1}\\
&-(n-2)((2n+1)sP_n - nP_{n-1})\\
=&(2n-1)((s^2-1)P'_n - nsP_n + nP_{n-1})
\end{aligned}
$$

which vanishes due to (5.15) as we wanted to show. ☐

Remark 5.2.1. In contrast with the case of the complete and path graphs, the polynomials $\{\mu^{\dagger}_{C_n}(s)\}_{n\in\mathbb{N}^+}$ do not conform an orthogonal family of polynomials. This can be checked noticing that the polynomials

$$
\mu^{\dagger}_3(s) = \frac{9s^2-1}{4}, \quad \mu^{\dagger}_4(s) = \frac{-9s^3+3s}{2}, \quad \mu^{\dagger}_5(s) = \frac{35s^4-20s^2+1}{4}
$$

do not satisfy a three-term recurrence of the form $\mu^{\dagger}_5(s) = (As+B)\mu^{\dagger}_4(s) - C\mu^{\dagger}_3(s)$, for constants A, B, C.

At this point we are ready to prove that μ-polynomials of the cyclic graphs C_n conform a Sturm sequence. This will be a consequence of Theorem 5.7 thanks to Theorem 5.4. We will include another proof on the real-rootedness and the simplicity of the roots in Appendix A.2.

Theorem 5.8. The family of polynomials $\mu_{C_n}(t)$ are real-rooted and forms an strictly interlacing sequence, that is $\mu_{C_n}(t)$ strictly interlaces $\mu_{C_{n+1}}(t)$ all $n \geq 1$, and hence the roots are also simple.

Proof. Letting $V_n(t) = (-1)^n\mu_{C_{n+1}}(t)$, deg $V_n = n$ and have nonnegative coefficients. The fact that V_n interlaces V_{n+1}, for $n = 0$ and 1 follows by direct inspection recalling that $V_0(t) = 1$, $V_1(t) = 1 + t$ and $V_2(t) = 2t^2 + 5t + 2$. Moreover, Theorem 5.7 asserts that V_n satisfies (5.7) with

$$
a_n = 2\frac{(n^2+n-1)t+n^2}{(n-1)(n+2)}, \quad b_n = \frac{2t(1-t)}{(n-1)(n+2)}, \quad c_n = -\frac{n(t-1)^2}{n+2},
$$

for $n \geq 2$ Since $c_n(t) < 0$ for all $t \neq 1$, the result follows from Theorem 5.4. ☐

We conclude this chapter with the real-rootedness of the μ-polynomials of the kite-like graphs discussed in Section 4.4.

Theorem 5.9. The family of polynomials $\mu_{U_n}(t)$ are real-rooted for $n \geq 1$. Moreover, $\mu_{P_n}(t)$ strictly interlaces $\mu_{U_{n+1}}(t)$ for all $n \geq 1$.

Proof. Recalling the formula (4.4) obtained in Example 4.4.1 and the recursion (5.8) for the Narayana polynomials, we see that

$$
\begin{aligned}
(n+1)|\mu|_{U_n}(t) =& 2(2n-1)(1+t)\mathcal{N}_{n-1}(t) - 2(n-2)(t-1)^2\mathcal{N}_{n-2}(t) - (n+1)t\mathcal{N}_{n-2}(t) \\
=& 2(2n-1)(1+t)\mathcal{N}_{n-1}(t) - (2(n-2)(t-1)^2 + (n+1)t)\mathcal{N}_{n-2}(t).
\end{aligned}
$$

The discriminant of the polynomial $-b(t) = 2(n-2)(t-1)^2 + (n+1)t$ is $-(n+1)(7n-17)$ which is negative for $n \geq 3$. Therefore, $-b(t) > 0$ for all $t \in \mathbb{R}$. It follows by Theorem 5.3 applied to $k = 1$ that $|\mu|_{U_n}(t)$ is real-rooted and that $\mathcal{N}_{n-1}(t) \prec |\mu|_{U_n}(t)$. This allows to conclude the proof. □

5.3 Comments on interlacing and contractions

We conclude this work with some comments on another interesting feature of μ-polynomials related to the interlacing property when contracting edges.

We have seen in the previous section the validity of Conjecture 3, namely,

$$\textit{Given a finite connected graph } G, \textit{there exists } e \in E(G) \textit{ such that } \mu_{G/e}(t) \prec \mu_G(t). \tag{5.19}$$

For the case of complete graphs K_n we saw in Example 5.1.2 that $\mu_{K_n}(t) \prec \mu_{K_n}(t)$. This can be interpreted as the previous statement, since any contraction by one edge of K_n produces K_{n-1}. The same is true for the family of path graphs P_n, cyclic graphs C_n, and star graphs St_n (see [12]), since any contraction by one edge leads to the corresponding graph on $[n-1]$. For the case of kite-like graphs U_n we were able to prove in Proposition 5.9 that $\mu_{P_{n-1}}(t) \prec \mu_{U_n}(t)$. Thus, (5.19) holds when we contract by any of the edges $\{n-2, n-1\}$, $\{n-2, n\}$ or $\{n-1, n\}$, see Figure 4.9. Numerical evidence has shown that $\mu_{U_{n-1}}(t) \prec \mu_{U_n}(t)$ also holds, but this is still an open question.

These results might lead to suppose that $\mu_{G/e}(t) \prec \mu_G(t)$, for any edge $e \in E(G)$. However, we have found a case that disprove this statement. We include the following example. Others can be obtained for graphs having the graph G of Example 5.3.1 as a subgraph. Therefore, the best we can conjecture is precisely (5.19).

Example 5.3.1. Consider the graph G on $[5]$ displayed in Figure 5.2 and its contraction G' by the edge $\{1, 3\}$. After some calculations we find that

$$
\begin{aligned}
\mu_G(t) &= 8t^4 + 65t^3 + 122t^2 + 65t + 8, \\
\mu_{G'}(t) &= -t^3 - 7t^2 - 7t - 1.
\end{aligned}
$$

The approximate value of the roots of $\mu_G(t)$ are

$$
\begin{aligned}
r_1 &\approx -5.69063128304352, \qquad r_2 \approx -1.65407307922546, \\
r_3 &\approx -0.604568209566811, \qquad r_4 \approx -0.175727428164204,
\end{aligned}
$$

while the values of the roots of $\mu_{G'}(t)$ are

$$s_1 = -3-2\sqrt{2} \approx -5.82842712474619, \quad s_2 = -1, \quad s_3 = -3+2\sqrt{2} \approx -0.171572875253810.$$

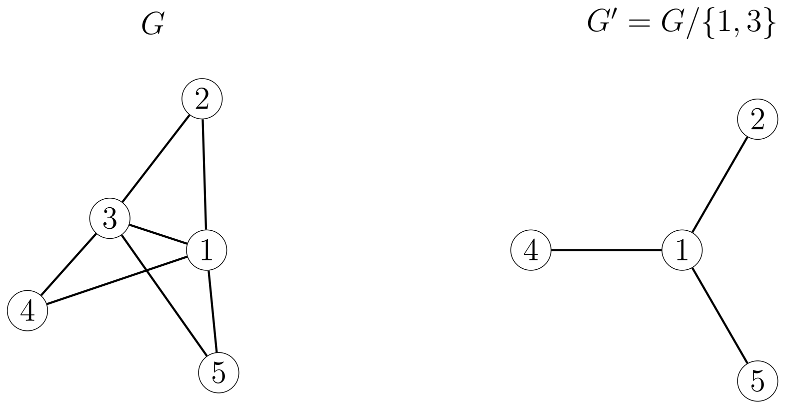

Figure 5.2: A contracted graph G' such that $\mu_{G'}(t) \not\prec \mu_G(t)$.

The curious order relation among these numbers is

$$s_1 < r_1 < r_2 < s_2 < r_3 < r_4 < s_3,$$

thus $\mu_{G'}(t) \not\prec \mu_G(t)$. This can be better visualized by looking at the transformed polynomials $\mu_G^{\dagger}$ and $\mu_{G'}^{\dagger}$, see Figure 5.3.

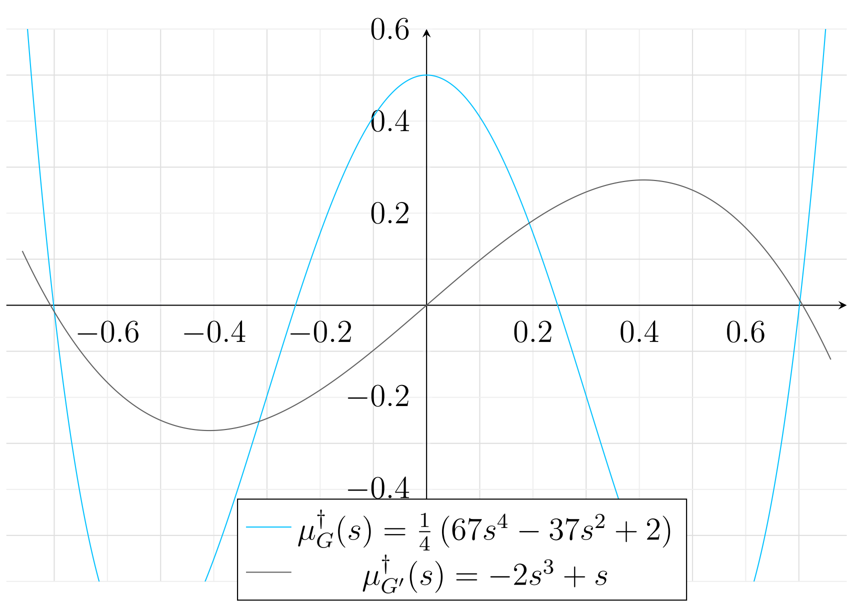

Figure 5.3

6

Future work

In this chapter we discuss three directions where the results of this project are being extended or simply directions that we are interested in pursuing next.

6.1 General building sets

The notion of connectedness is captured by the combinatorial structure of a building set. It turns out that the results that we have discussed in this work extend nicely to general building sets, which can also be thought as associated to the connectivity property in general hypergraphs. In the full article of this work we will include this more general perspective.

Let $\mathcal{B}$ be a building set on V. Generalizing the notion of bond partition of a graph we say that $\pi \in \Pi_V$ is a *bond partition* of a building set $\mathcal{B}$ if for every block $B \in \pi$ we have that $B \in \mathcal{B}$. We denote $\Pi_{\mathcal{B}}$ the poset of *building bond partitions* defined as the induced subposet of Π_V.

In [11], the authors extended their study of weighted partition posets to weighted bond posets $\mathcal{W}\Pi_G$ of graphs G. We extend further their definition to define the weighted bond poset of a building set $\mathcal{B}$.

A *weighted partition* of V is a set $\boldsymbol{\pi} = \{I_1^{\nu_1}, \dots, I_k^{\nu_k}\}$ where

- The *underlying partition* $\pi = \{I_1, \dots, I_k\} \in \Pi_V$.
- For all $i = 1, \dots, k$ we have $\nu_i \in \{0, 1, \dots, |I_i| - 1\}$.

The sets $I_i^{\nu_i}$ are called *weighted sets*.

The *poset of weighted partitions* $\mathcal{W}\Pi_V$ is the set of weighted partitions of V with an order relation defined for a pair

$$\{I_1^{\nu_1}, \dots, I_k^{\nu_k}\} \le \{J_1^{\mu_1}, \dots, J_l^{\mu_l}\}$$

whenever:

- $\{I_1, \dots, I_k\} \le \{J_1, \dots, J_l\}$ in Π_V
- if $J_r = I_{i_1} \cup \cdots \cup I_{i_s}$ then $\mu_r - (\nu_{i_1} + \cdots + \nu_{i_s}) \in \{0, 1, ..., s-1\}$.

For a building set $\mathcal{B}$ define the *weighted bond poset* $\mathcal{W}\Pi_{\mathcal{B}}$ as the induced subposet of $\mathcal{W}\Pi_V$ whose elements have underlying partitions in $\Pi_{\mathcal{B}}$.

The *μ-polynomial* of $\mathcal{B}$ is defined as the generating polynomial of the Möbius function of the maximal intervals of $\mathcal{W}\Pi_{\mathcal{B}}$, which are indexed by an integer $j = 0, \dots, n-1$, i.e.,

$$\mu_{\mathcal{B}}(t) := \sum_{j=0}^{n-1} \mu_{\mathcal{W}\Pi_{\mathcal{B}}}(\hat{0}, \{[n]^j\}) t^j.$$

It turns out that Theorems 2.1, 3.3 and 3.8 still hold for μ-polynomials of general building sets. These results will appear in the future article.

6.2 Real-rootedness

The evidence discussed and found in this work points to a general conjecture on the real-rootedness and the interlacing property both of the h and the μ polynomials of graphs.

The families of polynomials $h_{K_n}(t)$ [5] and $\mu_{K_n}(t)$ (see Example 5.1.2), $h_{P_n}(t)$ and $\mu_{P_n}(t)$ [5], and $h_{St_n}(t)$ and $\mu_{St_n}(t)$ [12] have been shown to be strictly interlacing. For the family of cycle graphs C_n, Theorem 5.8 shows that the polynomials $\mu_{C_n}(t)$ form a strictly interlacing sequence and the fact that $h_{C_n}(t) = W_{n-1}(t)$ are orthogonal polynomials (see Corollary 5.6) shows that $h_{C_n}(t)$ forms also an interlacing sequence. Theorem 5.9 also says that the kite-like graph U_{n+1} has the one-edge contraction P_n such that $\mu_{P_n}(t)$ strictly interlaces $\mu_{U_{n+1}}(t)$. Additional computational evidence for all graphs up to $n = 6$ vertices also supports the following conjecture. In fact, we can combine Conjectures 3 and 4 as follows.

Conjecture 5. For any connected graph G there are one-edge contractions G' and G'' of G that satisfies:

- $h_{G'}(t) \prec h_G(t)$,
- $\mu_{G''}(t) \prec \mu_G(t)$.

In particular, both $h_G(t)$ and $\mu_G(t)$ have only simple real roots.

6.3 γ-NONNEGATIVITY

An open problem from [11] that we have not addressed here is the question of γ-non negativity.

Since the h and μ-polynomials are symmetric or palindromic they can be written in the basis

$$\{t^i(1+t)^{d-2i} \mid i = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor\},$$

where d is the degree of the polynomial. It is a basic exercise in linear algebra (checking linear independence and dimension) to show that this is in fact a basis for the symmetric polynomials of degree d, which it is commonly refered to as the γ-basis. So a symmetric polynomial $f \in \mathbb{R}[t]$ of $\deg(f) = d$ can be written as

$$f(t) = \sum_{i=0}^{\left\lfloor \frac{d}{2} \right\rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

We say that f is γ-*nonnegative* if $\gamma_i \geq 0$ for all $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$.

In [16] the authors prove that $h_G(t)$ is γ-nonnegative for the family of chordal graphs G. In [11] the authors prove that $\mu_G(t)$ is also γ-nonnegative for any chordal graph G. The following conjecture is based on computational evidence in [11].

CONJECTURE 6 (GONZÁLEZ D'LEÓN - WACHS [11]). The polynomials $h_G(t)$ and $\mu_G(t)$ are γ-nonnegative for all graphs G.

A
Appendix

A.1 An inversion formula for an exponential function

The goal of this section is to give a proof of the identity

$$\left(\frac{e^{\alpha x}-e^{\beta x}}{(\alpha-\beta)e^{\alpha x}e^{\beta x}}\right)^{\circ(-1)} = \sum_{n=1}^{\infty}\left[\prod_{j=1}^{n-1}((n-j)\alpha+j\beta)\right]\frac{x^n}{n!} \tag{A.1}$$

valid for parameters $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$ [9, Example 1.7.2]. This will a consequence of

Theorem A.1 (Lagrange inversion theorem). Let $G(x) \in \mathbb{C}[[x]]$ a formal power series such that $G(0) = 0$ and $G'(0) \neq 0$. Then G has a compositional inverse which is given by

$$G^{\circ(-1)}(x) = \sum_{n=1}^{\infty} \operatorname{Res}\left(\frac{1}{G(x)^n}\right)\frac{x^n}{n}.$$

Here we recall that $\operatorname{Res}(B(x)) = b_{-1}$ denotes the coefficient corresponding to the power x^{-1}, for a formal Lauren series $B(x) = \sum_{n\in\mathbb{Z}} b_n x^n$.

For our proof we will also require the power series

$$\left(\frac{t}{e^t-1}\right)^n = \sum_{k=1}^{\infty} B_k^{(n)}\frac{t^k}{k!}, \qquad n \in \mathbb{N}^+. \tag{A.2}$$

The numbers $B_k^{(n)}$ are known as the *Nørlund polynomials*, which are in fact polynomials in n. More generally, we consider the series

$$\left(\frac{t}{e^t-1}\right)^n e^{xt} = \sum_{k=1}^{\infty} B_k^{(n)}(x)\frac{t^k}{k!}, \qquad n \in \mathbb{N}^+.$$

Multiplying the series (A.2) by the Taylor series of e^{xt} we find that

$$B_k^{(n)}(x) = \sum_{j=0}^{k} \binom{k}{j} B_j^{(n)} x^{k-j}.$$

These are known as the *generalized Bernoulli polynomials*. We recall the following fundamental recurrence for these polynomials.

PROPOSITION A.2. The generalizes Bernoulli polynomials satisfy the recurrence

$$B_k^{(n+1)}(x) = \left(1 - \frac{k}{n}\right) B_k^{(n)}(x) + \frac{k}{n}(x-n) B_{k-1}^{(n)}(x). \tag{A.3}$$

In particular,

$$B_n^{(n+1)}(x) = (x-1)(x-2)\cdots(x-n). \tag{A.4}$$

Proof. Note that the exponential generating series

$$\hat{f}(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$$

of a sequence $\{a_k\}_{k\in\mathbb{N}}$ satisfies

$$\hat{f}'(t) = \sum_{k=1}^{\infty} a_k \frac{t^{k-1}}{(k-1)!}, \qquad \frac{d}{dt}(tf) = \sum_{k=1}^{\infty} k a_{k-1} \frac{t^{k-1}}{(k-1)!} \qquad \frac{d}{dt}(t\hat{f}') = \sum_{k=1}^{\infty} k a_k \frac{t^{k-1}}{(k-1)!}.$$

Applying this to a_k given by the right hand side of (A.3) we find

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left[\left(1 - \frac{k}{n}\right) B_k^{(n)}(x) + \frac{k}{n}(x-n) B_{k-1}^{(n)}(x) \right] \frac{t^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} B_k^{(n)}(x) \frac{t^{k-1}}{(k-1)!} - \sum_{k=1}^{\infty} \frac{k}{n} B_k^{(n)}(x) \frac{t^{k-1}}{(k-1)!} + \frac{(x-n)}{n} \sum_{k=1}^{\infty} k B_{k-1}^{(n)}(x) \frac{t^{k-1}}{(k-1)!} \\
&= \frac{d}{dt} \left[\left(\frac{t}{e^t - 1}\right)^n e^{xt} - \frac{t}{n} \frac{d}{dt} \left[\left(\frac{t}{e^t-1}\right)^n e^{xt} \right] + \frac{t(x-n)}{n} \left(\frac{t}{e^t-1}\right)^n e^{xt} \right].
\end{aligned}$$

A brief calculations shows that

$$\left(\frac{t}{e^t-1}\right)^n e^{xt} \left[1 + \frac{t(x-n)}{n}\right] - \frac{t}{n}\frac{d}{dt}\left[\left(\frac{t}{e^t-1}\right)^n e^{xt}\right] = \left(\frac{t}{e^t-1}\right)^{n+1} e^{xt}.$$

Since

$$\frac{d}{dt}\left[\left(\frac{t}{e^t-1}\right)^{n+1} e^{xt}\right] = \sum_{k=1}^{\infty} B_k^{(n+1)} \frac{t^{k-1}}{(k-1)!},$$

equation (A.3) follows.

Finally, setting $k = n$ in (A.3), we find

$$B_n^{(n+1)}(x) = (x-n) B_{n-1}^{(n)}(x).$$

Proceeding inductively, we recover the closed expression (A.4) since $B_0^{(1)}(x) = 1$. □

Consider

$$F_\lambda(x) = \sum_{n=1}^{\infty} P_n(\lambda) \frac{x^n}{n!}$$

as the compositional inverse of the series

$$G_\lambda(x) = \frac{e^{\lambda x} - e^x}{(\lambda - 1)e^{\lambda x} e^x} = \frac{e^{-\lambda x} - e^{-x}}{1 - \lambda}.$$

This means that $F_\lambda(x)$ is the unique formal solution of the functional equation

$$e^{-\lambda F_\lambda(x)} - e^{-F_\lambda(x)} = (1 - \lambda)x. \tag{A.5}$$

The goal is to show that

$$P_n(\lambda) = \prod_{j=1}^{n-1} (j\lambda + n - j).$$

It is enough to prove (A.1) for $\beta = 1$. In general, if $G(x)$ and $H(x)$ are compositional inverses one of each other, then $G(ax)/a$ is the compositional inverse of $H(ax)/a$. Indeed, $G(a(H(ax)/a))/a = G(H(ax))/a = (ax)/a = x$. Therefore, taking $\lambda = \alpha/\beta$ we find

$$\frac{e^{\alpha x} - e^{\beta x}}{(\alpha - \beta)e^{\alpha x} e^{\beta x}} = \frac{1}{\beta} G_{\alpha/\beta}(\beta x)$$

has as compositional inverse

$$\frac{1}{\beta} F_{\alpha/\beta}(\beta x) = \sum_{n=1}^{\infty} \left[\prod_{j=1}^{n-1} ((n - j)(\alpha/\beta) + j) \right] \beta^{n-1} \frac{x^n}{n!},$$

which is precisely the right-hand side of (A.1).

To find $P_n(\lambda)$ we apply Lagrange inversion theorem.. Note that

$$\frac{x^n}{G_\lambda(x)^n} = \frac{((1 - \lambda)x)^n}{(e^{(1-\lambda)x} - 1)^n} e^{nx} = \sum_{k=1}^{\infty} B_k^{(n)} \left(\frac{n}{1 - \lambda} \right) (1 - \lambda)^k \frac{x^k}{k!}.$$

Therefore, selecting the $(n - 1)$th-coefficients of this series we find

$$\begin{aligned} \frac{P_n(\lambda)}{n!} = \frac{1}{n} \text{Res} \left(\frac{1}{G_\lambda(x)^n} \right) &= \frac{(1 - \lambda)^{n-1}}{n!} B_{n-1}^{(n)} \left(\frac{n}{1 - \lambda} \right) \\ &= \frac{(1 - \lambda)^{n-1}}{n!} \prod_{j=1}^{n-1} \left(\frac{n}{1 - \lambda} - j \right) = \frac{1}{n!} \prod_{j=1}^{n-1} (j\lambda + n - j). \end{aligned}$$

as it was required.

We conclude with some related formulas that can be deduced from (A.1).

1. If $\lambda = 0$, $P_n(0) = (n-1)!$. In this case $F_0(x) = (1-e^{-x})^{\circ(-1)} = -\log(1-x)$. Therefore,
$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{P_n(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

2. Replacing λ by $1/\lambda$ in (A.5) we see that
$$e^{-F_{1/\lambda}(x)/\lambda} - e^{-F_{1/\lambda}(x)} = \left(1 - \frac{1}{\lambda}\right) x.$$
Changing x by λx we find that
$$e^{-F_{1/\lambda}(\lambda x)} - e^{-F_{1/\lambda}(\lambda x)/\lambda} = (1-\lambda)\, x.$$
This means that $F_{1/\lambda}(\lambda x)/\lambda$ is a solution of (A.5). By uniqueness we conclude that
$$\frac{1}{\lambda} F_{1/\lambda}(\lambda x) = F_\lambda(x).$$
This means that
$$P_n(\lambda) = \lambda^{n-1} P_n(1/\lambda).$$
This is also clear from the explicit formula for $P_n(\lambda)$.

3. If we let $\lambda \to 1$, we see that $G_\lambda(x) \to xe^{-x}$. Therefore,
$$F_1(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} x^n$$
is the compositional inverse of $G_1(x) = xe^{-x}$. This recovers the fact that the W-Lambert series
$$W(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{n-1}}{n!} x^n$$
is the compositional inverse of xe^x near $x = 0$.

A.2 On the real-rootness of the $\mu(C_n)$-polynomials

Although a proof of the fact that the $\mu(C_n)$-polynomials have simple real roots was given via Theorem 5.4, we can give also a direct proof using more elementary tools. In fact, the following lines mimic the usual proof of Proposition 5.1, which can be seen in [2, Theorem 5.4.1].

Proof. Remark 5.1.1 shows that $\mu_{C_{n+1}}(t)$ is real-rooted with n simple zeros if and only if the same is true for
$$f_n(t) = (-1)^n \mu^{\dagger}_{C_{n+1}}(t) = \frac{2}{n+2} P'_{n+1}(t) - P_n(t) = \frac{1}{n+2} \left[nP_n(t) + 2tP'_n(t) \right], \qquad \text{(A.6)}$$

recall Proposition 5.5 and formula 5.13. Since $P_n(t)$ is real-rooted having simple zeros, the same is true for $P_n'(t)$ by Rolle's theorem. In fact, $P_n'(t)$ interlaces $P_n(t)$. We can thus apply Theorem 5.2 for $a = n$, $b = c = 0$, and $d = 2$ to conclude that $f_n(t)$ is real-rooted.

We prove now that $f_n(t)$ has simple zeros. First, notice that if $m < n - 2$, then

$$\int_{-1}^{1} t^m f_n(t)(1-t^2)dt = \int_{-1}^{1} t^m P_n^{(1,1)}(t)(1-t^2)dt - \int_{-1}^{1} P_n(t)(t^m - t^{m+2})dt = 0$$

due to the orthogonal relations of the Legendre and Jacobi polynomials.

Now, let m be the number of different roots of $f_n(t)$ which have odd multiplicity. If $\lambda_1, \ldots, \lambda_m$ are these roots, consider the polynomial

$$Q(t) = f_n(t)(x - \lambda_1) \cdots (x - \lambda_m).$$

Therefore, $Q(t) \geq 0$ for $t \in [-1, 1]$ since $Q(t)$ is the product of square powers of linear polynomials. If $m < n - 2$, the previous calculation implies that

$$\int_{-1}^{1} Q(t)(1-t^2)dt = 0,$$

contradicting that this integral is strictly positive. In this way, we find $m \geq n - 2$.

To prove that $m = n$, assume by contradiction that $m = n - 2$. Thus, $f_n(t)$ would have exactly $n - 2$ different roots with odd multiplicity, i.e., $n - 2$ simple roots. If ρ_0, ρ_1 are the remaining roots, both have even multiplicity, and this only can happen if $\rho_0 = \rho_1$. Moreover, since $\mu_{C_n}(t)$ is symmetric, we know from Remark 5.1.1 that $f_n(-t) = (-1)^n f_n(t)$. It follows that $-\rho_0$ is also a root of $f_n(t)$ with the same multiplicity as ρ_0. Being even, we also find that $-\rho_0 = \rho_0$, so $\rho_0 = 0$. Thus $f_n(0) = f_n'(0) = 0$. However, since

$$f_n'(t) = \frac{1}{n+2}\left[(n+2)P_n'(t) + 2tP_n''(t)\right],$$

it would follow that $P_n(0) = f_n(0) = 0$ and $P_n'(0) = f_n'(0) = 0$, contradicting that 0 is a simple root of P_n (which only happens when n is odd). In conclusion, $m = n$ and $f_n(t)$ has n simple real roots as required. □

A.3 Some codes

We include below three pseudo-codes we employed in this work. The complete codes used to compute the examples in this thesis were done in Python.

Algorithm 1 μ-polynomial of a graph G

Input: G a simple graph
Output: μ (μ-polynomial of G)

```
μ(𝒢)                                  ▷ set of graphs and their μ-polynomial previously calculated
function μPoly(G)
    if G is isomorphic to some graph in μ(𝒢) then
        (H, μ) ← the element in μ(𝒢) such that G ≅ H
    else
        if |V(G)| = 1 then
            μ ← 1
        else if G not is connected then
            μ ← 1
            for i ← 1 to k(G) do
                μ ← μ ∗ μPoly(G_i)            ▷ G_i is i-connected component of G
            end for
        else if G is connected then
            μ ← 0
            for π ∈ Π_G do
                p ← 1
                for B ∈ π do
                    p ← p ∗ μPoly(G|_B)
                end for
                μ ← μ − [|π|]_t ∗ p
            end for
        end if
        μ(𝒢) ← μ(𝒢) ∪ {(G, μ)}
    end if
    return μ
end function
```

Algorithm 2 Check v-admissibility of a partition π

Input: G a simple graph, π partition in Π_G and a vertex v of G
Output: TRUE or FALSE

```
function vAdmissible(G,v,π)
    for B ∈ π do
        a ← FALSE
        for u ∈ B do
            if {v,u} ∈ E(G) then
                a ← TRUE
                break
            end if
        end for
        if a = FALSE then
            return FALSE
        end if
    end for
    return TRUE
end function
```

Algorithm 3 μ-forest of a simple graph G

Input: G a simple graph
Output: $\mathcal{T}$ (μ-forest set of G)

function μFOREST(G)
  if $|V(G)| = 1$ **then**
    $\mathcal{T} \leftarrow \{(V(G), \emptyset)\}$
  else if G not is connected **then**
    $\mathcal{T} \leftarrow \emptyset$
    for $\mathcal{F} \in \mu\text{Forest}(G_1) \times \mu\text{Forest}(G_2) \times \cdots \times \mu\text{Forest}(G_{k(G)})$ **do**
      $V \leftarrow \bigcup_{T \in \mathcal{F}} V(T)$
      $E \leftarrow \bigcup_{T \in \mathcal{F}} E(T)$
      $\mathcal{T} \leftarrow \mathcal{T} \cup \{(V, E)\}$
    end for
  else
    $\mathcal{T} \leftarrow \emptyset$
    for $v \in V(G)$ **do**
      $H \leftarrow G|_{V(G) \setminus \{v\}}$
      for $\pi \in \Pi_H$ **do**
        if vADMISSIBLE(G, v, π) **then**
          $F \leftarrow \mu\text{FOREST}(H|_\pi)$
          $T \leftarrow \Big(V(G), E(F) \cup \bigcup_{r \in \text{Roots}(F)} \{(v, r)\}\Big)$
          $\mathcal{T} \leftarrow \mathcal{T} \cup \{T\}$
        end if
      end for
    end for
  end if
  return $\mathcal{T}$
end function

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